Biot-JKD's model : simulation of the propagation of poroelastic transient waves

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Motivations

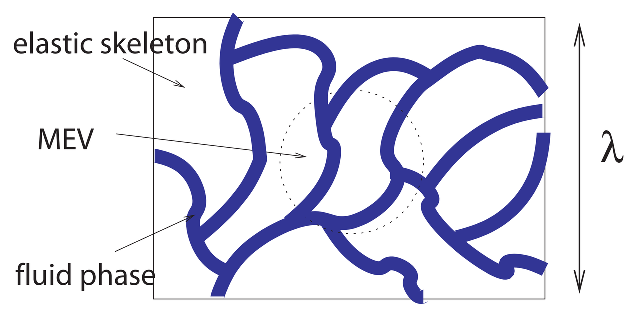
The propagation of waves in porous media has crucial implications in many areas, such as the characterization of industrial foams, spongious bones and petroleum rocks. We consider the Biot-JKD's model, which describes the dissipation terms in **high-frequency** (HF) for pores of random geometry with constant radius. It involves a frequency correction $\widehat{F}(\omega)$. An efficient **numerical modeling** in **time-domain** of the Biot-JKD's model remains a major challenge in real applications. The present algorithm can be generalized to:

- 2D
- space-varying parameters
- interfaces.

High-frequency modeling

Biot-JKD's model describes the propagation of mechanical waves in a porous medium consisting of a solid matrix saturated with fluid circulating freely through the pores. It is assumed that:

- $\lambda \gg$ diameter of the pores d
- small perturbations
- elastic and isotropic matrix saturated by a single fluid phase
- no thermal effect.



This model relies on 11 physical parameters: the density ρ_f and the dynamic viscosity η of the fluid; the density ρ_s and the shear modulus μ of the elastic skeleton; the porosity $0 < \phi < 1$, the tortuosity $a \geq 1$, the permeability κ_0, the Lamé coefficient λ_f, the two Biot's coefficients β and m of the saturated matrix, and the viscous characteristic length Λ. The unknowns are the elastic and acoustic displacements u_s and u_f, the elastic stress tensor σ, and the acoustic pressure p. In one hand, the constitutive laws are:

$$\begin{cases} \sigma = (\lambda_f + 2\,\mu)\,\varepsilon - m\,\beta\,\xi, \\ p = m\,(-\beta\ \mathrm{tr}\,\varepsilon + \xi)\,, \end{cases} \tag{1}$$

where ξ is the rate of fluid change, and ε is the strain tensor

$$\xi = -\frac{\partial}{\partial x}\left(\phi\,(u_f - u_s)\right), \qquad \varepsilon = \frac{\partial\,u_s}{\partial\,x}. \tag{2}$$

On the other hand, the conservation of momentum yields

$$\begin{cases} \rho\,\dfrac{\partial\,v_s}{\partial\,t} + \rho_f\,\dfrac{\partial\,w}{\partial\,t} = \dfrac{\partial\,\sigma}{\partial\,x}, \\ \rho_f\,\dfrac{\partial\,v_s}{\partial\,t} + \rho_w\,\dfrac{\partial\,w}{\partial\,t} + \dfrac{\eta}{\kappa_0}\,F(t) * w(t) = -\dfrac{\partial\,p}{\partial\,x}, \end{cases} \tag{3}$$

where $v_s = \frac{\partial u_s}{\partial t}$ is the elastic velocity, and $w = \phi\,\frac{\partial}{\partial t}\,(u_f - u_s)$ is the filtration velocity. The general Darcy's law (second equation of (3)) depends of the frequency regime via a correction $F(t)$. For the Biot-JKD's model, involving frequencies higher than

$$f_c = \frac{\eta\,\phi}{2\,\pi\,a\,\kappa\,\rho_f} = \frac{\omega_c}{2\,\pi}, \tag{4}$$

the frequency correction is:

$$\begin{aligned} \widehat{F}_{JKD}(\omega) &= \left(1 + i\,\omega\,\frac{4\,a^2\,\kappa_0^2\,\rho_f}{\eta\,\Lambda^2\,\phi^2}\right)^{1/2} \\ &= \left(1 + i\,P\,\frac{\omega}{\omega_c}\right)^{1/2} \\ &= \frac{1}{\sqrt{\Omega}}\,(\Omega + i\,\omega)^{1/2} \end{aligned} \tag{5}$$

where $\Omega = \frac{\omega_c}{P}$ and the Pride's number $P = \frac{4\,a\,\kappa_0}{\phi\,\Lambda^2} \simeq \frac{1}{2}$. This leads to:

$$F_{JKD}(t) * w(t) = \frac{1}{\sqrt{\Omega}}\,(D + \Omega)^{1/2} w(t). \tag{6}$$

The operator $(D + \Omega)^{1/2}$ is a **shifted fractional derivative**, defined by:

$$\begin{aligned} (D + \Omega)^{1/2} w(t) &\triangleq \frac{1}{\sqrt{\pi}}\,\frac{e^{-\Omega\,t}}{\sqrt{t}} * \left(\frac{\partial w}{\partial t}(t) + \Omega\,w(t)\right) \\ &= \mathcal{F}^{-1}\left((\Omega + i\,\omega)^{1/2}\,\widehat{w}(\omega)\right). \end{aligned} \tag{7}$$

Fractional derivative (7) generalizes the notion of classical derivative. It can be written with a convolution integral in time whose singular kernel is slowly decreasing and induces **memory effects**.

Diffusive reprensentation of the fractional derivative

We use a diffusive representation (DR) of the decreasing function $1/\sqrt{t}$ to rewrite the fractional derivative:

$$(D + \Omega)^{1/2} w(t) = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{\theta}}\,\psi(\theta, t)\,d\,\theta \approx \sum_{\ell=1}^{N} a_\ell\,\psi(\theta_\ell, t), \tag{8}$$

where N is the number of relaxation mechanisms, θ_ℓ are the relaxation frequencies and a_ℓ are the weights of the quadrature, and

$$\psi(\theta, t) = \int_0^t e^{-(\theta+\Omega)(t-\tau)}\left(\frac{\partial w}{\partial t}(\tau) + \Omega\,w(\tau)\right)\,d\tau \tag{9}$$

is solution of the **local-in-time** differential equation:

$$\begin{cases} \dfrac{\partial\psi}{\partial t} = -(\theta + \Omega)\,\psi + \left(\dfrac{\partial w}{\partial t} + \Omega\,w\right), \\ \psi(\theta, 0) = 0. \end{cases} \tag{10}$$

With this approximation, we define the Biot-DR's model, which is characterized by (1) and (3), with

$$\widehat{F}_{DR}(\omega) = \frac{\Omega + i\,\omega}{\sqrt{\Omega}}\sum_{\ell=1}^{N}\frac{a_\ell}{\theta_\ell + \Omega + i\,\omega}. \tag{11}$$

A velocity-stress formulation is obtained from (1), (3) and (10), leading to a first-order non-homogeneous linear system

$$\frac{\partial}{\partial\,t}\,\mathbf{U} + \mathbf{A}\,\frac{\partial}{\partial\,x}\,\mathbf{U} = -\mathbf{S}\,\mathbf{U}, \tag{12}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{S}$ are $(N+4)\times(N+4)$ real matrices depending on the physical parameters and the coefficients of the diffusive representation, and $\mathbf{U} = (v_s,\, w,\sigma\ p,\, \psi_1, ..., \psi_N)^T$ is the vector of unknows.

Determination of coefficients

We look for a function $\widehat{F}_{DR}(\omega)$ that approximates accurately $\widehat{F}_{JKD}(\omega)$ for $\omega_{min} \leq \omega \leq \omega_{max}$, where ω_{min} and ω_{max} obviously depend on the central frequency of the source f_0. We implement a classical linear least-squares minimization in the L_2 norm. Relaxation frequencies are distributed linearly on a logarithmic scale of N points:

$$\theta_\ell = \omega_{min}\left(\frac{\omega_{max}}{\omega_{min}}\right)^{\dfrac{\ell-1}{N-1}}, \qquad \ell = 1, ..., N. \tag{13}$$

The coefficients a_ℓ are then obtained by solving the linear system deduced from (11)

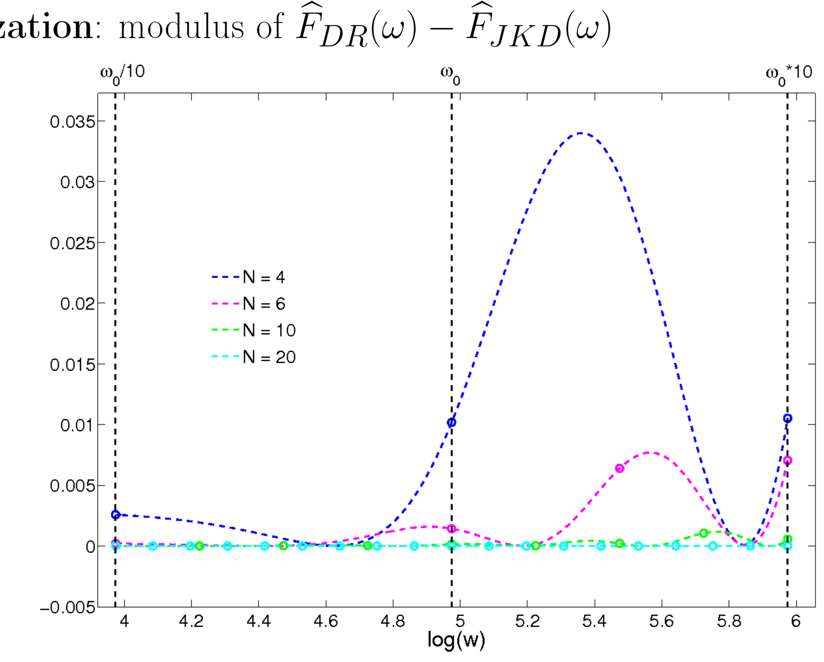
$$(\Omega + i\,\tilde{\omega}_k)^{1/2}\sum_{\ell=1}^{N}\frac{1}{\theta_\ell + \Omega + i\,\tilde{\omega}_k}\,a_\ell = 1, \qquad k = 1, ..., K, \tag{14}$$

where $\tilde{\omega}_k$ are distributed linearly on a logarithmic scale of K points:

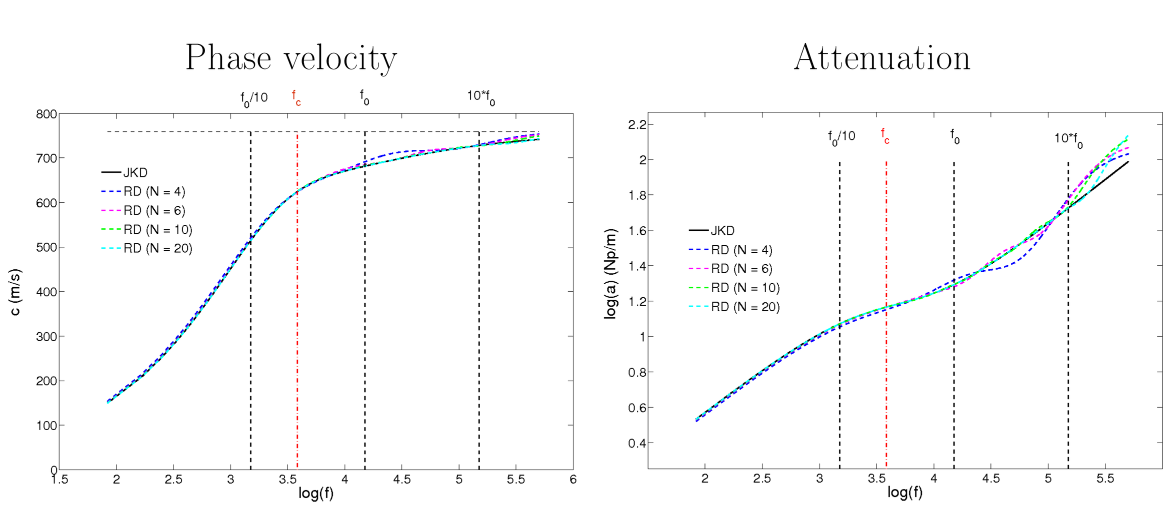
$$\tilde{\omega}_k = \omega_{min}\left(\frac{\omega_{max}}{\omega_{min}}\right)^{\dfrac{k-1}{K-1}}, \qquad k = 1, ..., K. \tag{15}$$

The angular frequencies θ_ℓ are all positive, but this is not true for the weights a_ℓ. In forthcoming numerical example, the medium is sandstone saturated with water ($f_c \simeq 4$ kHz). Waves are emitted by a ponctual stress source of finite duration at $x = 2$ m, of central frequency $f_0 = 15$ kHz.

Minimization: modulus of $\widehat{F}_{DR}(\omega) - \widehat{F}_{JKD}(\omega)$



Dispersion curves: focus on the slow compressional wave



Numerical method

To solve (12), we have developed an efficient numerical strategy, based on the following ingredients:

- `An explicit 4-th order ADER finite difference scheme`
  → very low numerical diffusion and dispertion rates,
  → stable under the optimal CFL condition.
- `A splitting method for integrating the source term in (12)`
  → exact integration of source term,
  → this integration of source term is stable if the real parts of the eigenvalues of $\mathbf{S}$ are positive, which can be proven whatever the coefficients of the diffusive representation obtained by the least-square method.

Numerous details and validations concerning the different algorithms are presented in [4, 5].

The total energy of the Biot-JKD's system decreases with time : the problem is well-posed. We show that this result is also true for the Biot-DR's system.

Numerical experiments

The numerical results are compared to the analytical solutions of the Biot-JKD's model (given by a Fourier analysis).

Snapshots at two instants

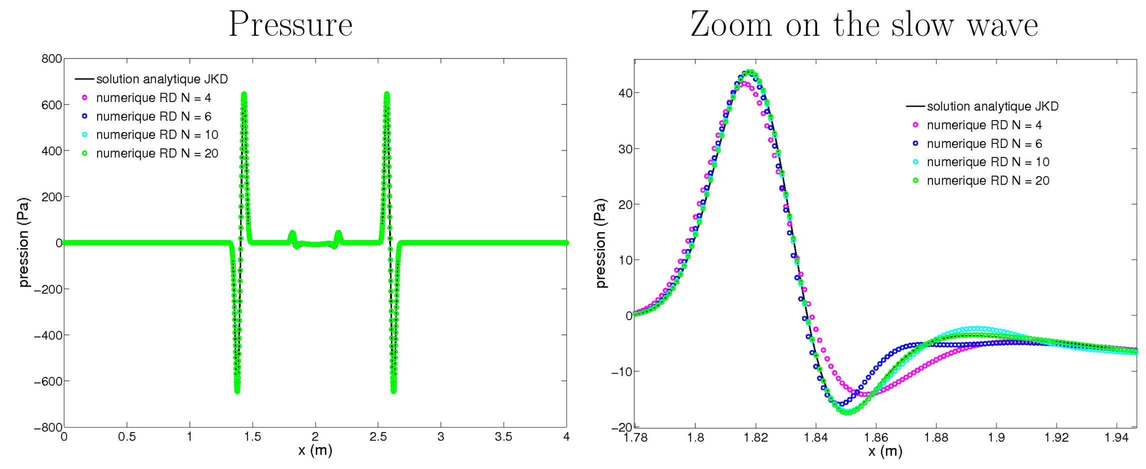


Figure 1: Pressure at $t = 0.28$ ms.

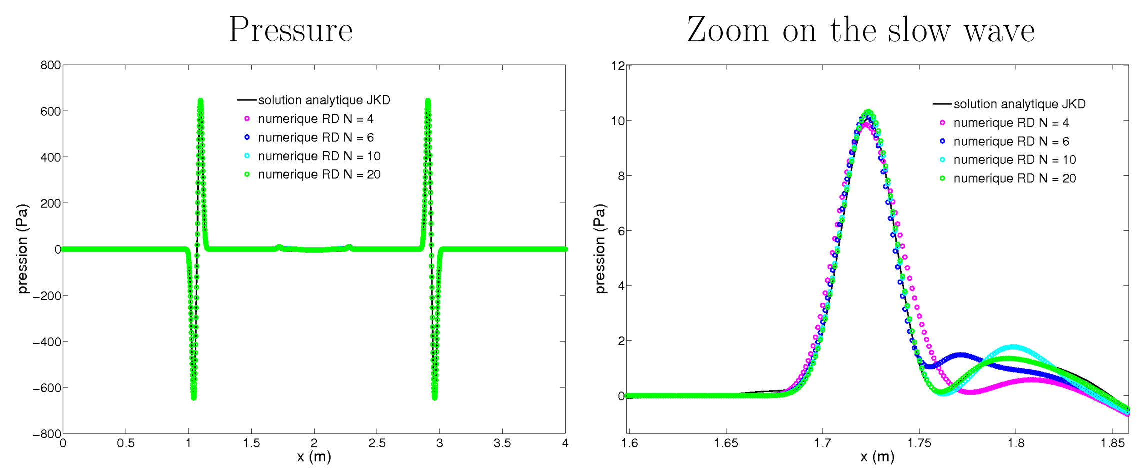


Figure 2: Pressure at $t = 0.42$ ms.

Increasing N clearly decreases the error between the Biot-DR's model and the Biot-JKD's model. Nevertheless taking a small number of mechanisms of relaxation ($N = 6$) leads to a numerical results that approximates very accurately the analytical solution of the Biot-JKD's model.

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