Time domain modeling of wave propagation in 2D acoustic / porous media

G. Chiavassa[1], B. Lombard[2], G. Lefeuve-Mesgouez[3], A. Mesgouez[3]

[1]Ecole Centrale Marseille and M2P2, [2]Laboratoire de Mécanique et d'Acoustique, Marseille, France, [3] Université d'Avignon et des pays de Vaucluse, Avignon, France

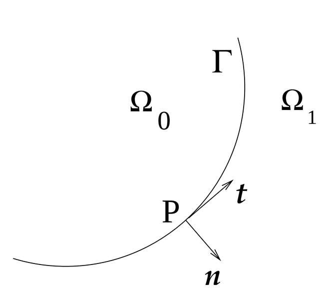
Motivations

The propagation of waves in porous media has crucial implications in many areas, such as the characterization of industrial foams, spongious bones and petroleum rocks. The most widely used model describing the propagation of mechanical waves in a saturated porous medium was proposed by Biot in 1956 [1]. A major achievement in **Biot's theory** was the prediction of a second (slow) compressional wave, besides the (fast) compressional wave and the shear wave classically propagated in elastic media.

In this work, a **numerical** method and a **semi-analytical** method are developed to investigate the wave propagation in 2D heterogeneous fluid / poroelastic media. Wave propagation is described by the usual acoustics equations (in the fluid medium) and by the low-frequency Biot's equations (in the porous medium). **Interface conditions** are introduced to model various hydraulic contacts between the two media: **open** pores, **sealed** pores, and **imperfect** pores.

Geometry

- Ω_0: fluid medium
- Ω_1: porous medium
- **t** and **n** represent tangential and normal vectors along the interface Γ



Physical modeling

Acoustics equations

In the fluid domain Ω_0, the physical parameters are the density ρ_f and the celerity of acoustic waves c. The acoustics equations write

$$\begin{cases} \rho_f \dfrac{\partial \mathbf{v}}{\partial t} + \nabla p = \mathbf{0}, \\ \dfrac{\partial p}{\partial t} + \rho_f \, c^2 \, \nabla.\mathbf{v} = f_p, \end{cases} \tag{1}$$

where $\mathbf{v}$ and p are the acoustic velocity and pressure and f_p the source term.

Low-frequency Biot equations

The poroelastic medium Ω_1 is modeled by the low-frequency Biot equations [2] where the physical parameters are

- the dynamic viscosity η and the density ρ_f of the saturating fluid.
- the density ρ_s and the shear modulus μ of the elastic skeleton;
- the porosity ϕ, the tortuosity $a > 1$, the absolute permeability κ, the Lamé coefficient of the saturated matrix λ_f, and the two Biot's coefficients β and m of the isotropic matrix.

The conservation of momentum and the constitutive laws yield

$$\begin{cases} \rho \dfrac{\partial \mathbf{v}_s}{\partial t} + \rho_f \dfrac{\partial \mathbf{w}}{\partial t} - \nabla.\sigma = \mathbf{0}, \\ \rho_f \dfrac{\partial \mathbf{v}_s}{\partial t} + \rho_w \dfrac{\partial \mathbf{w}}{\partial t} + \dfrac{\eta}{\kappa} \mathbf{w} + \nabla p = 0, \\ \sigma = \mathcal{C}\, \varepsilon(\mathbf{u_s}) - \beta \, \mathbf{p}\, \mathbf{I}, \\ p = -m \, (\beta \nabla.\mathbf{u}_s + \nabla.\mathcal{W}), \end{cases} \tag{2}$$

where $\mathbf{v}_s = \frac{\partial \mathbf{u}_s}{\partial t}$ is the elastic velocity, $\mathbf{w} = \phi(\mathbf{v}_f - \mathbf{v}_s) = \frac{\partial \mathcal{W}}{\partial t}$ is the filtration velocity, $\mathbf{v}_f$ is the fluid velocity, σ is the elastic stress tensor, $\varepsilon(\mathbf{u_s}) = \frac{1}{2}\left(\nabla \mathbf{u_s} + ^{\mathbf{T}} \nabla \mathbf{u_s}\right)$ is the elastic strain tensor, and p is the pressure.

To be valid, the system (2) requires that the spectrum of the waves lies mainly in the low-frequency range, involving frequencies lower than

$$f_c = \frac{\eta \, \phi}{2 \, \pi \, a \, \kappa \, \rho_f}. \tag{3}$$

Interface conditions

Following [2] the general conditions are assumed at the interface Γ:

$$\begin{cases} \mathbf{v}_0.\mathbf{n} = \mathbf{v}_{s1}.\mathbf{n} + \mathbf{w}_1.\mathbf{n}, \\ -p_0 \, \mathbf{n} = \sigma_\mathbf{1}.\mathbf{n}, \\ [p] = -\dfrac{1}{\mathcal{K}} \dfrac{\mathbf{w}_1.\mathbf{n}}{|\mathbf{n}|}. \end{cases} \tag{4}$$

Depending on the value of $\mathcal{K}$, the hydraulic permeability of the interface, various limit-cases are encountered:

- **open pores**: if $\mathcal{K} \to +\infty$.
- **sealed pores**: if $\mathcal{K} \to 0$.
- **imperfect pores**: if $0 < \mathcal{K} < +\infty$.

Numerical method

In both media, a velocity-stress formulation is obtained from (1) and (2) leading to a first-order non-homogeneous linear system

$$\frac{\partial}{\partial t}\mathbf{U} + \mathbf{A}\frac{\partial}{\partial x}\mathbf{U} + \mathbf{B}\frac{\partial}{\partial y}\mathbf{U} = -\mathbf{S}\,\mathbf{U}. \tag{5}$$

$\mathbf{A}$, $\mathbf{B}$ are 3×3 real matrices in Ω_0 and 8×8 in Ω_1. $\mathbf{S} = 0$ in the fluid domain. Mathematical and dispersion analysis of this system in Ω_1 point out the following properties:

- The spectral radius of $\mathbf{S}$ is large for realistic porous media
  → **stiff system** leading to inefficient numerical time steps
- The Biot's slow wave behaves like a diffusive mode with a very low phase velocity → presence of **small localized spatial scales**

To overcome these difficulties, we have developed an efficient numerical strategy, based on the following ingredients:

1) `An explicit 4-th order ADER Finite Difference Scheme`
   → very low numerical diffusion and dispertion rates.
2) `A Splitting method to deal with the source term in (5)`
   → exact integration of source term effect to recover the optimal CFL stability condition associated to homogeneous equation.
3) `A local Space-Time mesh refinement`
   → high accuracy around interfaces to represent the spatial and temporal evolution of the slow wave.
4) `An Immersed Interface Method`
   → conservation of a 4-th order scheme at interfaces, accurate representation of the interface geometry, introduction of the interface conditions.

Numerous details and validations of the different algorithms are presented in [3, 4].

Analytical method

The analytical approach deals with a fluid overlying horizontal poroelastic layers. **Green's functions** for the pressure and the velocity of a specific observation point are derived using the following steps: *i)* writting of the wave equations relative to the fluid part and those relative to the porous layers, *ii)* Fourier transforms, on both time and horizontal space variables, *iii)* solution to the Helmholtz equation relative to the fluid part, *iv)* exact stiffness matrix approach for the solution of wave equations relative to the porous layers with a specific attention to the conditionning of the matrices [5], *v)* assembling of the layer matrices and the contributions due to the interface conditions between the porous medium and the fluid. The resulting system can be summarized as follows

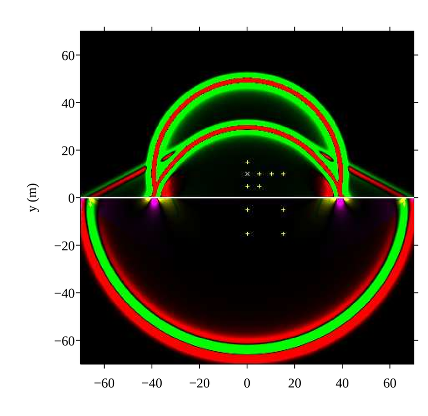
$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots \; T_{22} - \dfrac{i\rho_f\omega^2}{k_{3f}} & T_{23} - \dfrac{i\rho_f\omega^2}{k_{3f}} & & \dots \\ \dots \; T_{32} - \dfrac{i\rho_f\omega^2}{k_{3f}} & T_{33} - \dfrac{i\rho_f\omega^2}{k_{3f}} - \dfrac{i\omega}{\mathcal{K}} & & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} \overline{u}^*_{s1|x_3=0} \\ i\overline{u}^*_{s3|x_3=0} \\ i\overline{\mathcal{W}}^*_{3|x_3=0} \\ \dots \\ \dots \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{S(\omega)}{2\pi k_{3f}} e^{ik_{3f}x_{3s}} \\ \dfrac{S(\omega)}{2\pi k_{3f}} e^{ik_{3f}x_{3s}} \\ 0 \\ \dots \end{Bmatrix} \tag{6}$$

where k_{3f} is the fluid wavenumber, $S(\omega)$ the source spectrum, x_{3s} the source altitude and T the layer stiffness matrix depending on the compressional and shear wavenumbers and on the Fourier transform parameters.

Then, the resulting system is solved to obtain either the pression or the velocity in the frequency-wavenumber domain. The inverse Fourier transform is performed by using an adaptive Filon type quadrature for the wavenumber integration. This procedure is particularly adapted for **rapid oscillatory integrands** as occuring here.

Numerical Experiments

Plane interface: A plane interface separates water (upper half) and a porous medium (lower half) corresponding to sintered glass beads [6]. The source is located in the water just above the interface and its time history is of Ricker type with a central frequency of 103.65 kHz.



Numerical simulation is done with a global spatial mesh size of $4.10^{-4} m$ refined by a factor 5 around the interface. Time history of the vertical velocity (v_{s2}) just below the interface is presented on figures 2 and 3.

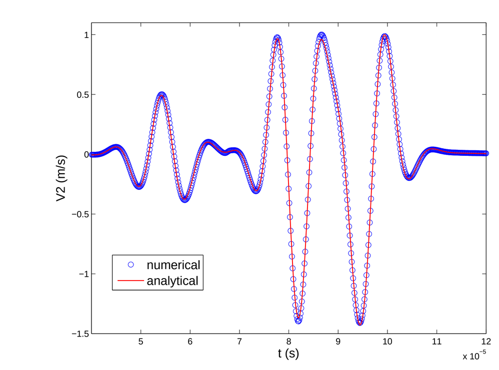


FIGURE 1: **Open pores**

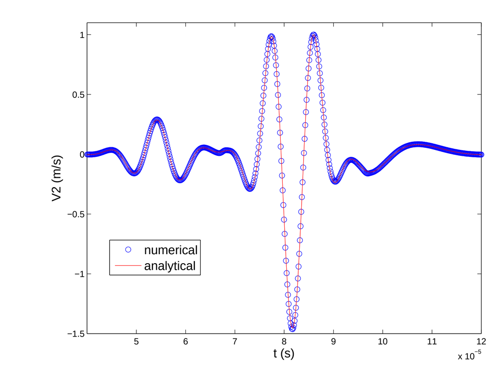


FIGURE 2: **Sealed pores**

→ Excellent agreement is obtain between numerical and analytical methods

→ Surface waves are highly sensitive to the surface contact

Sinusoidal interface: Poroelastic medium (lower half) corresponds here to unconsolidated sand [6]. Mesh size is $2m$ except around the interface where it is refined by a factor 7. A Ricker source of central frequency 20 Hz ($\sim f_c/63$) is used.

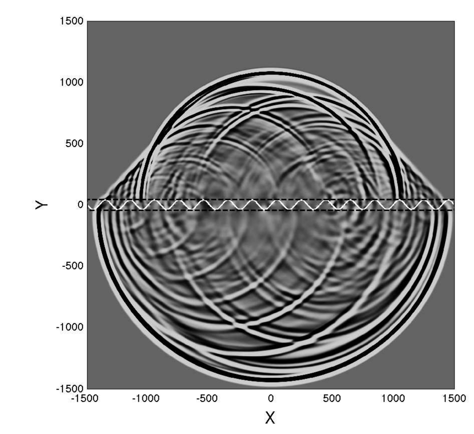


FIGURE 3: Snapshot of the pressure after 800 iterations for imperfect hydraulic contact ($\mathcal{K} = 5.10^{-7} m.s^{-1}.Pa^{-1}$)

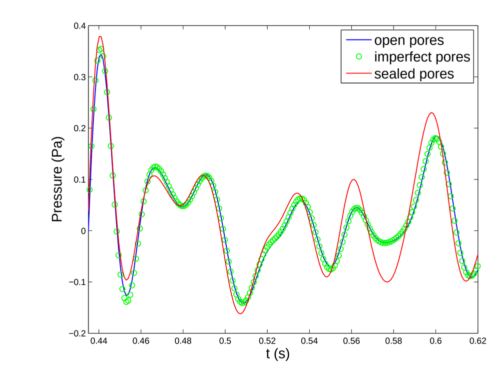
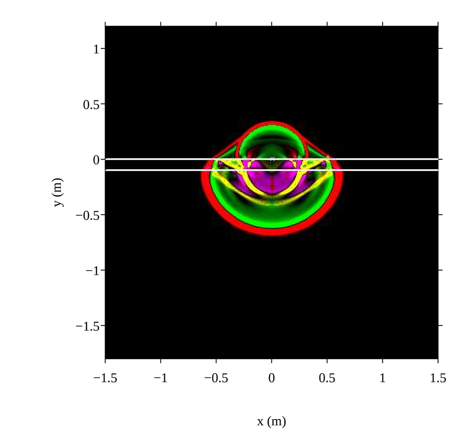
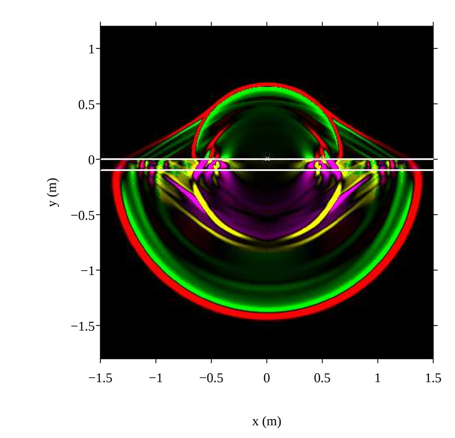


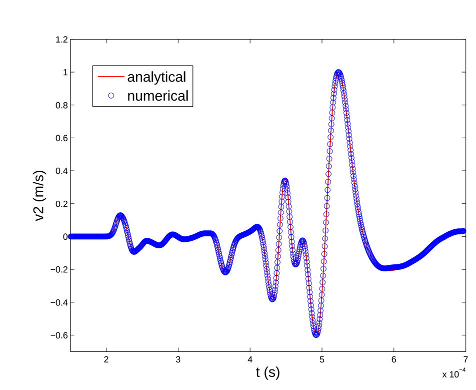
FIGURE 4: Time history of p recorded in water close to the interface

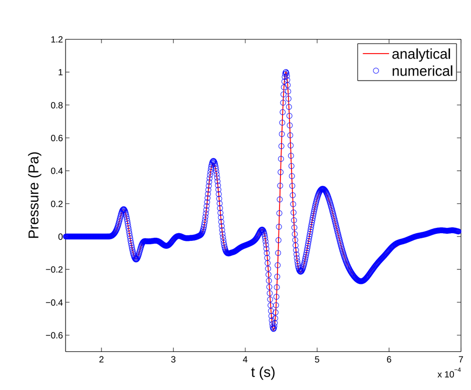
Multiple interfaces: A Ricker type source of central frequency 20 kHz located in water generates compressional waves propagating through two different porous media. Snapshot of pressure at two different times:





Time history of vertical velocity and pressure recorded at the water interface for sealed pores conditions:





References

[1] M. A. Biot, *Theory of propagation of elastic waves in a fluid-saturated porous solid. I: Low-frequency range*, J. Acoust. Soc. Am., 28-2 (1956), 168-178.

[2] T. Bourbié, O. Coussy, B. Zinszner, *Acoustics of Porous Media*, Gulf Publishing Company (1987).

[3] G. Chiavassa, B. Lombard and J. Piraux: *Numerical modeling of 1-D transient poroelastic waves in the low-frequency range*, J. Comput. Appl. Math., Vol. 234, 6, (2010).

[4] G. Chiavassa, B. Lombard: *Time domain numerical modeling of wave propagation in 2D heterogeneous porous media*, J. Comp. Phys, Vol. 230 (2011).

[5] A. Mesgouez, G. Lefeuve-Mesgouez: *Transient solution for multilayered poroviscoelastic media obtained by an exact stiffness matrix formulation* Int. J. Numer. Anal. Meth. Geomech., Vol. 33, 1911-1931, (2009).

[6] R. Sidler, J. M. Carcione, K. Holliger: *Simulation of surface waves in porous media*, Geophys. J. Int., 183-2 (2010).