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On cylindrical waves in anisotropic Cartesian materials and its implications on the harmonic mode separability assumption

Martin Östberg

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Purpose

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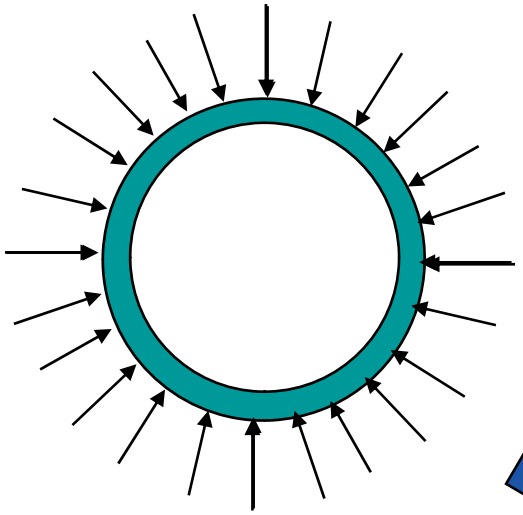
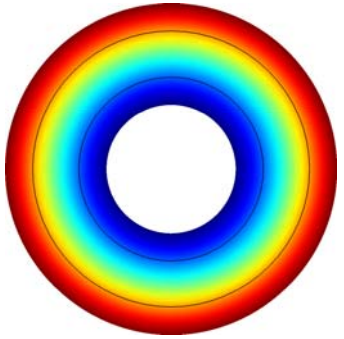
Load

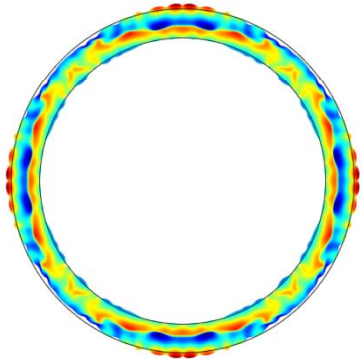


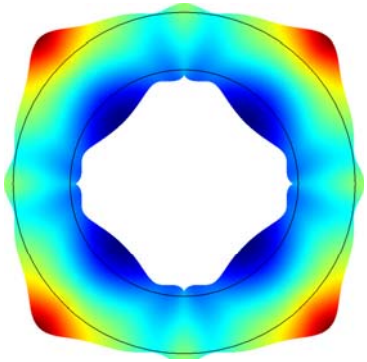
Response

- Rotational symmetric anisotropic structures (elastic, porous, fluid)
  - Orthogonal modes 3D→2D



- Non aligned anisotropy
  - Break down of mode assumption, ”mode-spreading”





Rotationally symmetric multilayered structures

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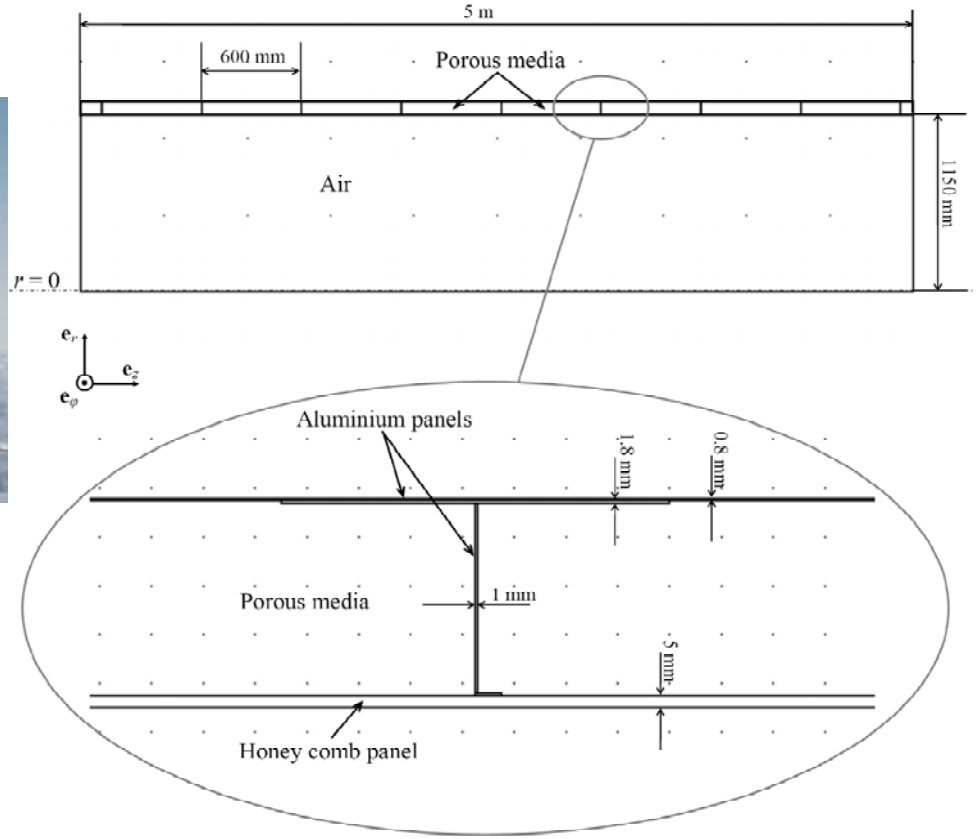
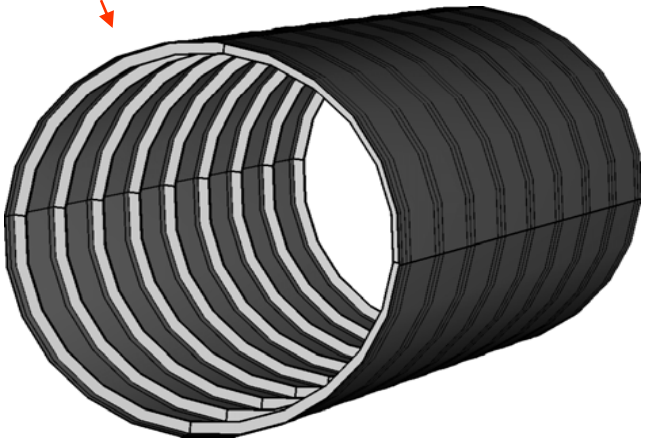


Acoustic load



Computationally costly FE-models

Semianalytical methods.

Previous work

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- Hörlin N-E, Göransson P. Weak, anisotropic symmetric formulations of Biot's equations for vibro-acoustic modelling of porous elastic materials. *International Journal of Numerical Methods in Engineering*, 2010; **84**: 1519–1540 .

- Östberg M, Hörlin N-E, Göransson P. Weak formulation of Biot's equations in cylindrical coordinates with harmonic expansion in the circumferential direction, *International Journal for Numerical Methods in Engineering*, 2010; **81**: 1439-1454.

- Östberg M, Göransson P, Hörlin N-E, Kari L. Weak forms for modelling of rotationally symmetric, multilayered structures, including anisotropic poro-elastic media, *International Journal for Numerical Methods in Engineering,* In press

Waves in anisotropic porous media

Homogenised two-phase continuum

Constitutive equations

$$\boldsymbol{\sigma}^{\mathrm{s}} = \left(\mathbb{C} + \frac{\mathbf{Q} \otimes \mathbf{Q}}{R} \right) : \boldsymbol{\varepsilon} + \mathbf{Q}\theta^{\mathrm{f}}$$

$$\boldsymbol{\sigma}^{\mathrm{f}} = -\phi p \mathbf{I} = \left(\mathbf{Q} : \boldsymbol{\varepsilon} + R\theta^{\mathrm{f}} \right) \mathbf{I}$$

Momentum equations

$$\nabla \cdot \boldsymbol{\sigma}^{\mathrm{s}} = -\omega^2 \tilde{\boldsymbol{\rho}}^{11} \mathbf{u}^{\mathrm{s}} - \omega^2 \tilde{\boldsymbol{\rho}}^{12} \mathbf{u}^{\mathrm{f}}$$

$$\nabla \cdot \boldsymbol{\sigma}^{\mathrm{f}} = -\omega^2 \tilde{\boldsymbol{\rho}}^{12} \mathbf{u}^{\mathrm{s}} - \omega^2 \tilde{\boldsymbol{\rho}}^{22} \mathbf{u}^{\mathrm{f}}$$

$\boldsymbol{\varepsilon}$: solid strain

$\mathbf{u}^{\mathrm{f}}$: fluid displacement

$\mathbf{u}^{\mathrm{s}}$: solid displacement

$\boldsymbol{\sigma}^{\mathrm{f}}$: fluid stress

$\boldsymbol{\sigma}^{\mathrm{s}}$: solid stress

$$\theta^{\mathrm{f}} = \nabla \cdot \mathbf{u}^{\mathrm{f}}$$

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Waves in porous media (Biot)

”Helmholtz” equation (waves in fluids

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Waves in porous media (Biot)

”Helmholtz” equation (waves in fluids

”Naviers” equation (waves in solids)

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”Helmholtz” equation (waves in fluids

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Constitutive equations

$$\boldsymbol{\sigma}^{\mathrm{s}} = \left(\mathbb{C} + \frac{\mathbf{Q}\otimes\mathbf{Q}}{R}\right) : \boldsymbol{\varepsilon} + \mathbf{Q}\theta^{\mathrm{f}}$$

$$\boldsymbol{\sigma}^{\mathrm{f}} = -\phi p\mathbf{I} = \left(\mathbf{Q} : \boldsymbol{\varepsilon} + R\theta^{\mathrm{f}}\right)\mathbf{I}$$

Coupling terms

$\boldsymbol{\varepsilon}$: solid strain

$\mathbf{u}^{\mathrm{f}}$: fluid displacement

$\mathbf{u}^{\mathrm{s}}$: solid displacement

$\boldsymbol{\sigma}^{\mathrm{f}}$: fluid stress

$\boldsymbol{\sigma}^{\mathrm{s}}$: solid stress

$$\theta^{\mathrm{f}} = \nabla\cdot\mathbf{u}^{\mathrm{f}}$$

Momentum equations

$$\nabla\cdot\boldsymbol{\sigma}^{\mathrm{s}} = -\omega^2\tilde{\boldsymbol{\rho}}^{11}\mathbf{u}^{\mathrm{s}} - \omega^2\tilde{\boldsymbol{\rho}}^{12}\mathbf{u}^{\mathrm{f}}$$

$$\nabla\cdot\boldsymbol{\sigma}^{\mathrm{f}} = -\omega^2\tilde{\boldsymbol{\rho}}^{12}\mathbf{u}^{\mathrm{s}} - \omega^2\tilde{\boldsymbol{\rho}}^{22}\mathbf{u}^{\mathrm{f}}$$

Anisotropy, elastic part

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- Transersely Isotropic
  (5 params)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} A & B & C & 0 & 0 & 0 \\ & A & C & 0 & 0 & 0 \\ & & D & 0 & 0 & 0 \\ & & & (A-B)/2 & 0 & 0 \\ symm. & & & & E & 0 \\ & & & & & E \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \end{bmatrix}$$

- Orthotropic
  (9 params)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ symm. & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \end{bmatrix}$$

- Fully Anisotropic
  (21 params)

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ symm. & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \end{bmatrix}$$

Anisotropy

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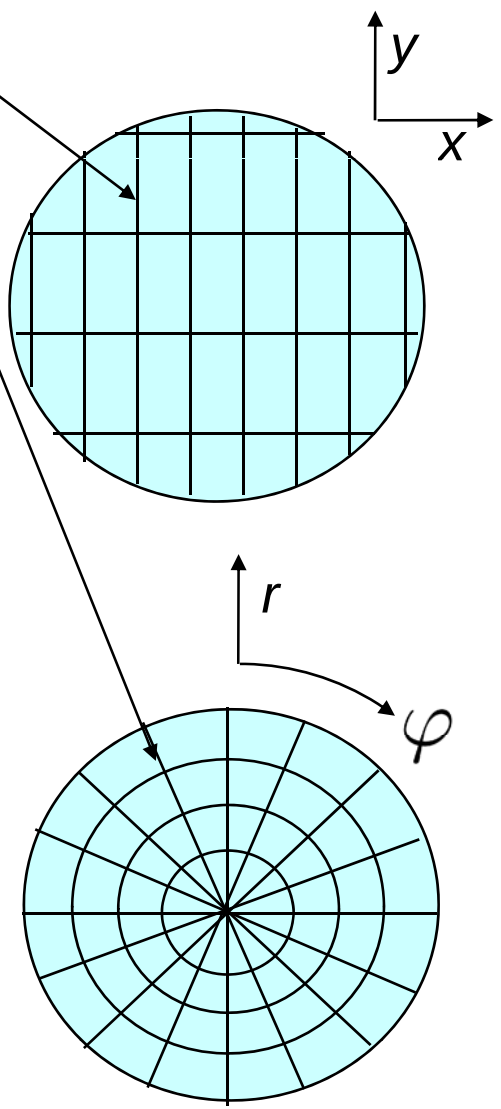
- Constant in Cartesian

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} C_{xxxx} & C_{xxyy} & C_{xxzz} & 0 & 0 & 0 \\ & C_{yyyy} & C_{yyzz} & 0 & 0 & 0 \\ & & C_{zzzz} & 0 & 0 & 0 \\ & & & C_{xyxy} & 0 & 0 \\ & & & & C_{yzyz} & 0 \\ symm. & & & & & C_{xzxz} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{xy} \\ 2\epsilon_{yz} \\ 2\epsilon_{xz} \end{bmatrix}$$

- Constant in cylindrical

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\varphi\varphi} \\ \sigma_{zz} \\ \sigma_{r\varphi} \\ \sigma_{\varphi z} \\ \sigma_{rz} \end{bmatrix} = \begin{bmatrix} C_{rrrr} & C_{xxyy} & C_{rrzz} & 0 & 0 & 0 \\ & C_{\varphi\varphi\varphi\varphi} & C_{\varphi\varphi zz} & 0 & 0 & 0 \\ & & C_{zzzz} & 0 & 0 & 0 \\ & & & C_{r\varphi r\varphi} & 0 & 0 \\ & & & & C_{\varphi z\varphi z} & 0 \\ symm. & & & & & C_{rzrz} \end{bmatrix} \begin{bmatrix} \epsilon_{rr} \\ \epsilon_{\varphi\varphi} \\ \epsilon_{zz} \\ 2\epsilon_{r\varphi} \\ 2\epsilon_{\varphi z} \\ 2\epsilon_{rz} \end{bmatrix}$$

Principal directions



Material constant in cyl. coordinates

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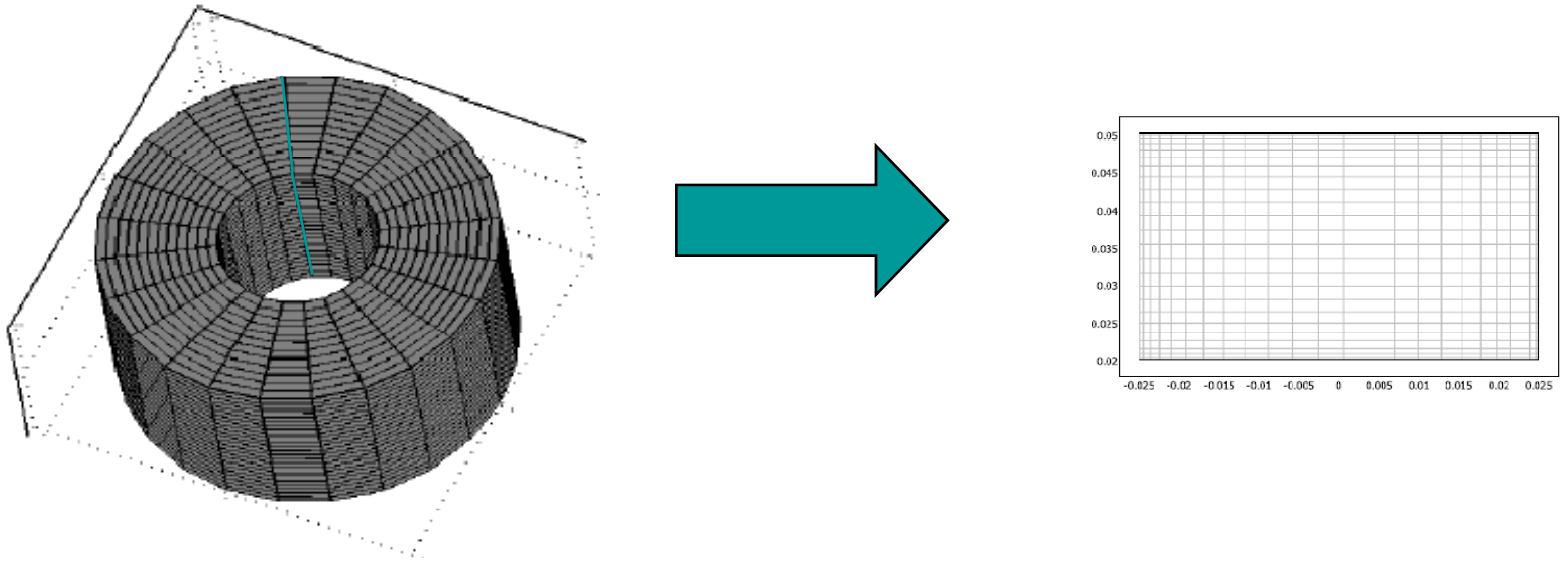
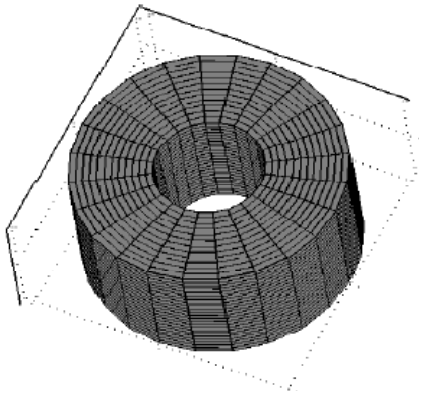
- Method for reducing 3D problems
  - Fourier expansion + FEM → 2D problems
- Without loss of generality or accuracy!



Material constant in cyl. coordinates



- Elastic wave Eq. (Navier)
  - Strong form:

$$\nabla[\mathbf{C}(\nabla^{\mathrm{T}}\mathbf{U})] = -\rho\omega^2\mathbf{U}$$

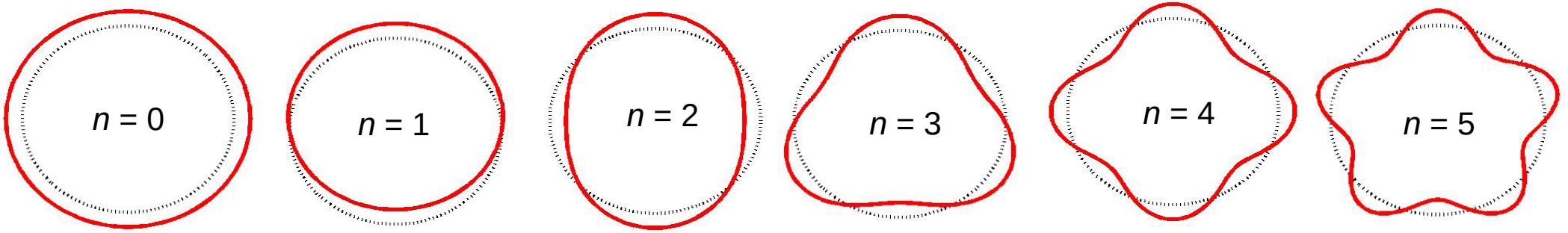
  - Weak form (3D):

$$\int_{\Omega} [\mathbf{C}(\nabla^s)^{\mathrm{T}}\mathbf{U}]^{\mathrm{T}}(\nabla^s)^{\mathrm{T}}\mathbf{V}\mathrm{d}\Omega - \int_{\Omega} \omega^2\rho\mathbf{U}^{\mathrm{T}}\mathbf{V}\mathrm{d}\Omega = \int_{\Gamma} (\mathbf{T}^{\mathrm{tot}})^{\mathrm{T}}\mathbf{V}\mathrm{d}\Gamma$$

  - Fourier ansatz:

$$\mathbf{U}(r,\varphi,z) = \sum_{n=-\infty}^{\infty} \mathbf{u}_n(r,z)\mathrm{e}^{\mathrm{i}n\varphi}$$

$$\mathbf{V}(r,\varphi,z) = \mathbf{V}(r,z)\mathrm{e}^{\mathrm{i}m\varphi}$$



Material constant in cyl. coordinates

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- Orthogonality

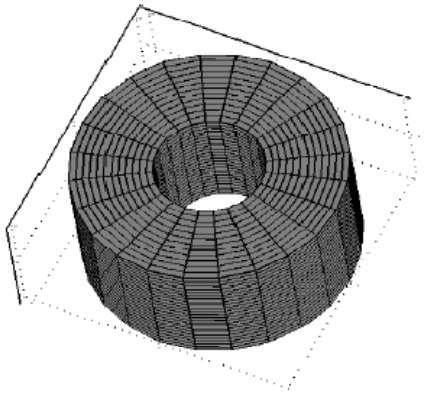
$$\int pq^* r \,\mathrm{d}r \,\mathrm{d}\varphi \,\mathrm{d}z = \sum_{n=-\infty}^{\infty} \int P_n Q r \,\mathrm{d}r \,\mathrm{d}z \int \mathrm{e}^{\mathrm{i}(m-n)\varphi} \,\mathrm{d}\varphi = 2\pi \int P_m Q r \,\mathrm{d}r \,\mathrm{d}z$$

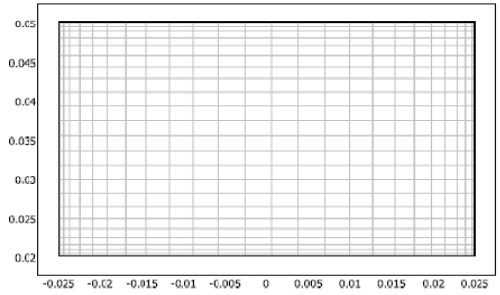
- Angular dependence eliminated → 2D problems!

$$\int_\Omega \left[\mathbf{c}(\nabla_n)^{\mathrm{T}}\mathbf{u}_n\right]^{\mathrm{T}} (\nabla_n)^{\mathrm{T}} \mathbf{v} \mathrm{d}\Omega - \int_\Omega \omega^2 \rho \mathbf{u}_n^{\mathrm{T}} \mathbf{v} \mathrm{d}\Omega = \int_\Gamma (\mathbf{t}_n^{\mathrm{tot}})^{\mathrm{T}} \mathbf{v} \mathrm{d}\Gamma$$

- Solution is obtained: $\mathbf{U}(r,\varphi,z) = \sum_{n=-\infty}^{\infty} \mathbf{u}_n(r,z)\mathrm{e}^{\mathrm{i}n\varphi}$

$\mathbf{U}(r,\varphi,z)$

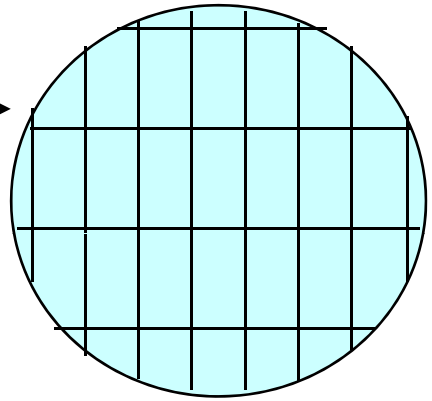




$\mathbf{u}_n(r,z)$

Anisotropy; Coordinate transformation

Constant in Cart. coords.



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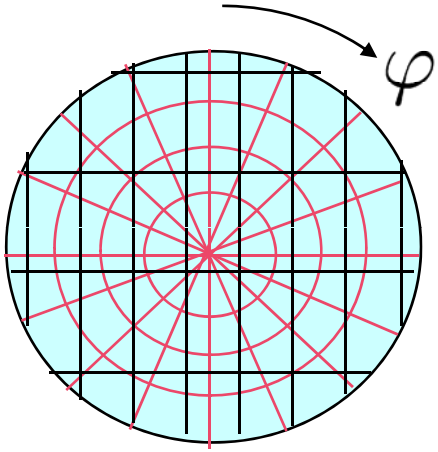
Rotation

$$[\mathbf{R}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_{\gamma_1\alpha_1} R_{\gamma_2\alpha_2} \ldots R_{\gamma_n\alpha_n} \Theta_{\gamma_1\gamma_2\ldots\gamma_n} = \bar{\Theta}_{\alpha_1,\alpha_2,\ldots,\alpha_n}$$

Orthotropic Cartesian

Cylindrical

$$[\mathbb{C}] = \begin{bmatrix} C_{xxxx} & C_{xxyy} & C_{xxzz} & 0 & 0 & 0 \\ & C_{yyyy} & C_{yyzz} & 0 & 0 & 0 \\ & & C_{zzzz} & 0 & 0 & 0 \\ & & & C_{xyxy} & 0 & 0 \\ symm. & & & & C_{xzxz} & 0 \\ & & & & & C_{yzyz} \end{bmatrix}$$

Sparse, spatially indep.

$$[\mathbb{C}] = \begin{bmatrix} C_{rrrr} & C_{rr\varphi\varphi} & C_{rrzz} & C_{rrr\varphi} & C_{rrrz} & C_{rr\varphi z} \\ & C_{\varphi\varphi\varphi\varphi} & C_{\varphi\varphi zz} & C_{\varphi\varphi r\varphi} & C_{\varphi\varphi rz} & C_{\varphi\varphi\varphi z} \\ & & C_{zzzz} & C_{zzr\varphi} & C_{zzrz} & C_{zz\varphi z} \\ & & & C_{r\varphi r\varphi} & C_{r\varphi rz} & C_{r\varphi\varphi z} \\ symm. & & & & C_{rzrz} & C_{rz\varphi z} \\ & & & & & C_{\varphi z\varphi z} \end{bmatrix}$$

Full, spatially dep. of φ

Effects of anisotropy

Material params. constant in Cart. coords.

expressed in cyl coords:

$$R_{\gamma_1\alpha_1}R_{\gamma_2\alpha_2}...R_{\gamma_n\alpha_n}\Theta_{\gamma_1\gamma_2...\gamma_n} = \bar{\Theta}_{\alpha_1,\alpha_2,...,\alpha_n}$$

No. of modes = tensor order

$$[\mathbb{C}] = [\mathbb{C}_0] + [\mathbb{C}_1] \times \mathrm{e}^{\mathrm{i}\varphi} + [\mathbb{C}_{-1}] \times \mathrm{e}^{-\mathrm{i}\varphi} + ... + [\mathbb{C}_{-4}] \times \mathrm{e}^{-4\mathrm{i}\varphi}$$

Elastic wave Eq., cyl. coords.

φ- dependent C:

$$\sum_{k=-4}^{4}\sum_{n=-\infty}^{\infty}\int_{\Omega}\mathrm{e}^{\mathrm{i}(k+n-m)}\left[\mathbf{c}_k(\nabla)^{\mathrm{T}}\mathbf{u}_n\right]^{\mathrm{T}}(\nabla)^{\mathrm{T}}\mathbf{v}_m\mathrm{d}\Omega -$$

$$\sum_{n=-\infty}^{\infty}\int_{\Omega}\omega^2\rho\mathbf{u}_n^{\mathrm{T}}\mathrm{e}^{\mathrm{i}(n-m)}\mathbf{v}_m\mathrm{d}\Omega = \sum_{n=-\infty}^{\infty}\int_{\Gamma}(\mathbf{t}_n^{\mathrm{tot}})^{\mathrm{T}}\mathrm{e}^{\mathrm{i}(n-m)}\mathbf{v}_m\mathrm{d}\Gamma$$

Constant C:

$$\int_{\Omega}\left[\mathbf{c}(\nabla_n)^{\mathrm{T}}\mathbf{u}_n\right]^{\mathrm{T}}(\nabla_n)^{\mathrm{T}}\mathbf{v}\mathrm{d}\Omega - \int_{\Omega}\omega^2\rho\mathbf{u}_n^{\mathrm{T}}\mathbf{v}\mathrm{d}\Omega = \int_{\Gamma}(\mathbf{t}_n^{\mathrm{tot}})^{\mathrm{T}}\mathbf{v}\mathrm{d}\Gamma$$

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Effects of anisotropy

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$$\sum_{k=-4}^{4}\sum_{n=-\infty}^{\infty}\int_{\Omega}\overbrace{\mathrm{e}^{\mathrm{i}(k+n-m)}}^{\delta_{k(m-n)}}\left[\mathbf{c}_k(\nabla)^{\mathrm{T}}\mathbf{u}_n\right]^{\mathrm{T}}(\nabla)^{\mathrm{T}}\mathbf{v}_m\mathrm{d}\Omega-$$

$$\sum_{n=-\infty}^{\infty}\int_{\Omega}\omega^2\rho\mathbf{u}_n^{\mathrm{T}}\overbrace{\mathrm{e}^{\mathrm{i}(n-m)}}^{\delta_{mn}}\mathbf{v}_m\mathrm{d}\Omega=\sum_{n=-\infty}^{\infty}\int_{\Gamma}(\mathbf{t}_n^{\mathrm{tot}})^{\mathrm{T}}\overbrace{\mathrm{e}^{\mathrm{i}(n-m)}}^{\delta_{mn}}\mathbf{v}_m\mathrm{d}\Gamma$$

$$\sum_{n=-\infty}^{\infty}\int_{\Omega}\underbrace{\left[\mathbf{c}_{m-n}\right.}\underbrace{\left.(\nabla_n^s)^{\mathrm{T}}\mathbf{u}_n\right]^{\mathrm{T}}}(\nabla_{-m}^s)^{\mathrm{T}}\mathbf{v}_m\mathrm{d}\Omega-\int_{\Omega}\omega^2\rho\mathbf{u}_m^{\mathrm{T}}\mathbf{v}_m\mathrm{d}\Omega=\int_{\Gamma}\underbrace{(\mathbf{t}_m^{\mathrm{tot}})^{\mathrm{T}}}\mathbf{v}_m\mathrm{d}\Gamma$$

Hooke's tensor relates load of order *m* to response of order *n*

Response of order *n*

Load of order *m*

Spatially invariant media $\mathbf{c}_{m-n}=\delta_{mn}\mathbf{c}_0$

Modes are not independent!!!

Rotated Hookes tensor structure

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$$[\mathbb{C}_0] = \begin{bmatrix} A_0 & B_0 & C_0 & 0 & 0 & 0 \\ & A_0 & C_0 & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & (A_0 - B_0)/2 & 0 & 0 \\ & & & & E_0 & 0 \\ symm. & & & & & E_0 \end{bmatrix} \quad [\mathbb{C}_1^+] = \begin{bmatrix} 0 & 0 & 0 & 0 & A_1 + B_1 & \mathrm{i}(A_1 - B_1) \\ & 0 & 0 & 0 & A_1 - B_1 & \mathrm{i}(A_1 + B_1) \\ & & 0 & 0 & C_1 & \mathrm{i}C_1 \\ & & & 0 & -\mathrm{i}B_1 & -B_1 \\ symm. & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

$$[\mathbb{C}_2^+] = \begin{bmatrix} A_2 & 0 & B_2 & -\mathrm{i}A_2/2 & 0 & 0 \\ & -A_2 & -B_2 & -\mathrm{i}A_2/2 & 0 & 0 \\ & & 0 & -\mathrm{i}B_2 & 0 & 0 \\ & & & 0 & 0 & 0 \\ symm. & & & & E_2 & \mathrm{i}E_2 \\ & & & & & -E_2 \end{bmatrix} \quad [\mathbb{C}_3^+] = \begin{bmatrix} 0 & 0 & 0 & 0 & A_3 & \mathrm{i}A_3 \\ & 0 & 0 & 0 & -A_3 & -\mathrm{i}A_3 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & -\mathrm{i}A_3 & A_3 \\ symm. & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

Hooke's tensor comps.

$$[\mathbb{C}] = [\mathbb{C}_0] + [\mathbb{C}_1] \times \mathrm{e}^{\mathrm{i}\varphi} + [\mathbb{C}_{-1}] \times \mathrm{e}^{-\mathrm{i}\varphi} + ... + [\mathbb{C}_{-4}] \times \mathrm{e}^{-4\mathrm{i}\varphi}$$

$$[\mathbb{C}_4^+] = \begin{bmatrix} A_4 & -A_4 & 0 & -\mathrm{i}A_4 & 0 & 0 \\ & A_4 & 0 & \mathrm{i}A_4 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & -A_4 & 0 & 0 \\ symm. & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}^{00} & \mathbf{M}^{01} & \mathbf{M}^{02} & \mathbf{M}^{03} & \mathbf{M}^{04} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{M}^{10} & \mathbf{M}^{11} & \mathbf{M}^{12} & \mathbf{M}^{13} & \mathbf{M}^{14} & \mathbf{M}^{15} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{M}^{20} & \mathbf{M}^{21} & \mathbf{M}^{22} & \mathbf{M}^{23} & \mathbf{M}^{24} & \dots & \mathbf{M}^{26} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{M}^{30} & \mathbf{M}^{31} & \mathbf{M}^{32} & \mathbf{M}^{33} & \mathbf{M}^{34} & \dots & \dots & \mathbf{M}^{37} & \dots & \mathbf{0} \\ \mathbf{M}^{40} & \mathbf{M}^{41} & \mathbf{M}^{42} & \mathbf{M}^{43} & \dots & \dots & \dots & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{51} & \mathbf{M}^{52} & \mathbf{M}^{53} & \dots & \dots & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{M}^{(M-1)(N-5)} & \dots & \mathbf{M}^{(M-1)(N-1)} & \mathbf{M}^{(M-1)N} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{M}^{M(N-4)} & \dots & \mathbf{M}^{M(N-1)} & \mathbf{M}^{MN} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{U}}_0 \\ \hat{\mathbf{U}}_1 \\ \hat{\mathbf{U}}_2 \\ \hat{\mathbf{U}}_3 \\ \hat{\mathbf{U}}_4 \\ \hat{\mathbf{U}}_5 \\ \hat{\mathbf{U}}_6 \\ \hat{\mathbf{U}}_7 \\ \hat{\mathbf{U}}_8 \\ \vdots \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{F}}_0 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

Discretized system

Rotated Hookes tensor structure

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$$[\mathbb{C}_0] = \begin{bmatrix} A_0 & B_0 & C_0 & 0 & 0 & 0 \\ & A_0 & C_0 & 0 & 0 & 0 \\ & & C_3 & 0 & 0 & 0 \\ & & & (A_0 - B_0)/2 & 0 & 0 \\ & & & & E_0 & 0 \\ \textit{symm.} & & & & & E_0 \end{bmatrix}$$

$$[\mathbb{C}_1^+] = \begin{bmatrix} 0 & 0 & 0 & 0 & A_1 + B_1 & \mathrm{i}(A_1 - B_1) \\ & 0 & 0 & 0 & A_1 - B_1 & \mathrm{i}(A_1 + B_1) \\ & & 0 & 0 & C_1 & \mathrm{i}C_1 \\ & & & 0 & -\mathrm{i}B_1 & -B_1 \\ \textit{symm.} & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

$$[\mathbb{C}_2^+] = \begin{bmatrix} A_2 & 0 & B_2 & -\mathrm{i}A_2/2 & 0 & 0 \\ & -A_2 & -B_2 & -\mathrm{i}A_2/2 & 0 & 0 \\ & & 0 & -\mathrm{i}B_2 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \textit{symm.} & & & & E_2 & \mathrm{i}E_2 \\ & & & & & -E_2 \end{bmatrix}$$

$$[\mathbb{C}_3^+] = \begin{bmatrix} 0 & 0 & 0 & 0 & A_3 & \mathrm{i}A_3 \\ & 0 & 0 & 0 & -A_3 & -\mathrm{i}A_3 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & -\mathrm{i}A_3 & A_3 \\ \textit{symm.} & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

Hooke's tensor comps.

Neglect axial (z) dependence

$$[\mathbb{C}_4^+] = \begin{bmatrix} A_4 & -A_4 & 0 & -\mathrm{i}A_4 & 0 & 0 \\ & A_4 & 0 & \mathrm{i}A_4 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & -A_4 & 0 & 0 \\ \textit{symm.} & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}^{00} & \mathbf{M}^{01} & \mathbf{M}^{02} & \mathbf{M}^{03} & \mathbf{M}^{04} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{M}^{10} & \mathbf{M}^{11} & \mathbf{M}^{12} & \mathbf{M}^{13} & \mathbf{M}^{14} & \mathbf{M}^{15} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{M}^{20} & \mathbf{M}^{21} & \mathbf{M}^{22} & \mathbf{M}^{23} & \mathbf{M}^{24} & \dots & \mathbf{M}^{26} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{M}^{30} & \mathbf{M}^{31} & \mathbf{M}^{32} & \mathbf{M}^{33} & \mathbf{M}^{34} & \dots & \dots & \mathbf{M}^{37} & \dots & \mathbf{0} \\ \mathbf{M}^{40} & \mathbf{M}^{41} & \mathbf{M}^{42} & \mathbf{M}^{43} & \dots & \dots & \dots & \dots & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{51} & \mathbf{M}^{52} & \mathbf{M}^{53} & \dots & \dots & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{M}^{(M-1)(N-5)} & \dots & \mathbf{M}^{(M-1)(N-1)} & \mathbf{M}^{(M-1)N} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{M}^{M(N-4)} & \dots & \mathbf{M}^{M(N-1)} & \mathbf{M}^{MN} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{U}}_0 \\ \hat{\mathbf{U}}_1 \\ \hat{\mathbf{U}}_2 \\ \hat{\mathbf{U}}_3 \\ \hat{\mathbf{U}}_4 \\ \hat{\mathbf{U}}_5 \\ \hat{\mathbf{U}}_6 \\ \hat{\mathbf{U}}_7 \\ \hat{\mathbf{U}}_8 \\ \vdots \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{F}}_0 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

Discretized system

Rotated Hookes tensor structure

$$[\mathbb{C}_0] = \begin{bmatrix} A_0 & B_0 & C_0 & 0 & 0 & 0 \\ & A_0 & C_0 & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & (A_0 - B_0)/2 & 0 & 0 \\ symm. & & & & E_0 & 0 \\ & & & & & E_0 \end{bmatrix}$$

$$[\mathbb{C}_1^+] = \begin{bmatrix} 0 & 0 & 0 & 0 & A_1 + B_1 & \mathrm{i}(A_1 - B_1) \\ & 0 & 0 & 0 & A_1 - B_1 & \mathrm{i}(A_1 + B_1) \\ & & 0 & 0 & C_1 & \mathrm{i}C_1 \\ & & & 0 & -\mathrm{i}B_1 & -B_1 \\ symm. & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

$$[\mathbb{C}_2^+] = \begin{bmatrix} A_2 & 0 & B_2 & -\mathrm{i}A_2/2 & 0 & 0 \\ & -A_2 & -B_2 & -\mathrm{i}A_2/2 & 0 & 0 \\ & & 0 & -\mathrm{i}B_2 & 0 & 0 \\ & & & 0 & 0 & 0 \\ symm. & & & & E_2 & \mathrm{i}E_2 \\ & & & & & -E_2 \end{bmatrix}$$

$$[\mathbb{C}_3^+] = \begin{bmatrix} 0 & 0 & 0 & 0 & A_3 & \mathrm{i}A_3 \\ & 0 & 0 & 0 & -A_3 & -\mathrm{i}A_3 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & -\mathrm{i}A_3 & A_3 \\ symm. & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

Hooke's tensor comps.

Neglect axial (z) dependence

$$[\mathbb{C}_4^+] = \begin{bmatrix} A_4 & -A_4 & 0 & -\mathrm{i}A_4 & 0 & 0 \\ & A_4 & 0 & \mathrm{i}A_4 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & -A_4 & 0 & 0 \\ symm. & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix}$$

Cxxxx=Cyyyy

$$\begin{bmatrix} \mathbf{M}^{00} & \mathbf{M}^{01} & \mathbf{M}^{02} & \mathbf{M}^{03} & \mathbf{M}^{04} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{M}^{10} & \mathbf{M}^{11} & \mathbf{M}^{12} & \mathbf{M}^{13} & \mathbf{M}^{14} & \mathbf{M}^{15} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{M}^{20} & \mathbf{M}^{21} & \mathbf{M}^{22} & \mathbf{M}^{23} & \mathbf{M}^{24} & \cdots & \mathbf{M}^{26} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{M}^{30} & \mathbf{M}^{31} & \mathbf{M}^{32} & \mathbf{M}^{33} & \mathbf{M}^{34} & \cdots & \cdots & \mathbf{M}^{37} & \cdots & \mathbf{0} \\ \mathbf{M}^{40} & \mathbf{M}^{41} & \mathbf{M}^{42} & \mathbf{M}^{43} & \cdots & \cdots & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{M}^{51} & \mathbf{M}^{52} & \mathbf{M}^{53} & \cdots & \cdots & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{M}^{(M-1)(N-5)} & \cdots & \mathbf{M}^{(M-1)(N-1)} & \mathbf{M}^{(M-1)N} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{M}^{M(N-4)} & \cdots & \mathbf{M}^{M(N-1)} & \mathbf{M}^{MN} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{U}}_0 \\ \hat{\mathbf{U}}_1 \\ \hat{\mathbf{U}}_2 \\ \hat{\mathbf{U}}_3 \\ \hat{\mathbf{U}}_4 \\ \hat{\mathbf{U}}_5 \\ \hat{\mathbf{U}}_6 \\ \hat{\mathbf{U}}_7 \\ \hat{\mathbf{U}}_8 \\ \vdots \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{F}}_0 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

Discretized system

Modelling example

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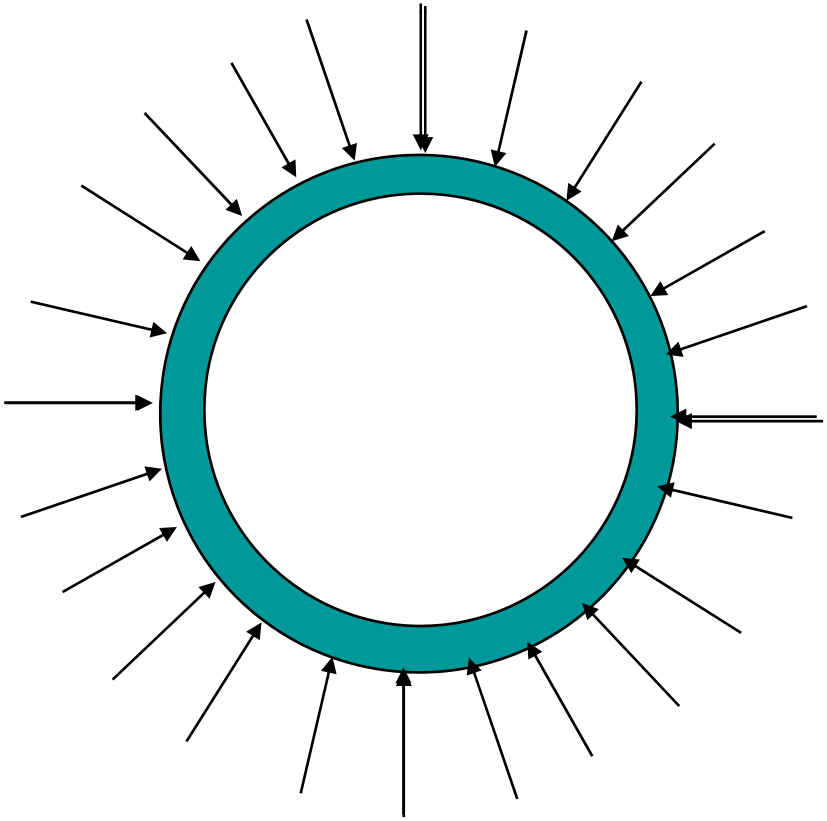

Inner radius: 0.5 m
Outer radius: 0.6 m

Homogenous (zero-order) load

Plane strain (no z-dep)

Orthotropic media

$$\begin{bmatrix} 5 & 0.5 & C_{xz} & 0 & 0 & 0 \\ & 5 & C_{yz} & 0 & 0 & 0 \\ & & C_{zz} & 0 & 0 & 0 \\ & & & (5-0.5)2 & 0 & 0 \\ symm. & & & & C_{yzyz} & 0 \\ & & & & & C_{xzxz} \end{bmatrix} \times 10^5 \text{Pa}$$

Modelling example

Mode no. 0

Mode no. 4

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Mode no. 64

Modelling example

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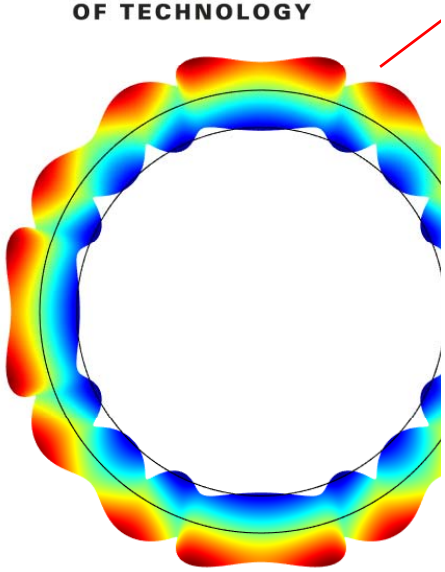

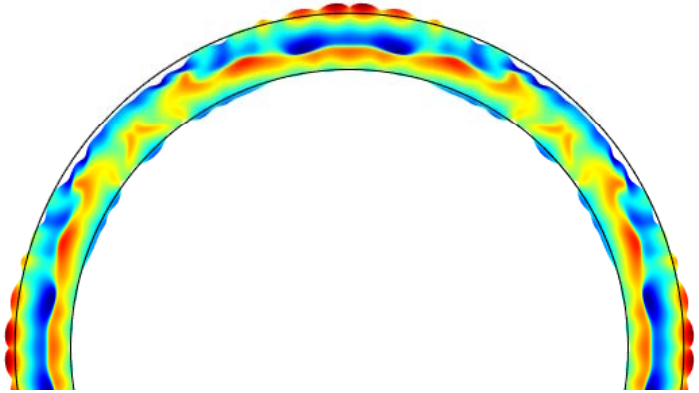

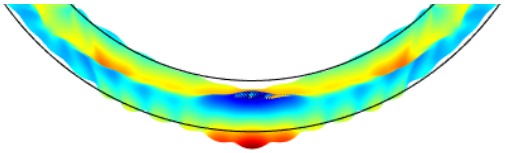

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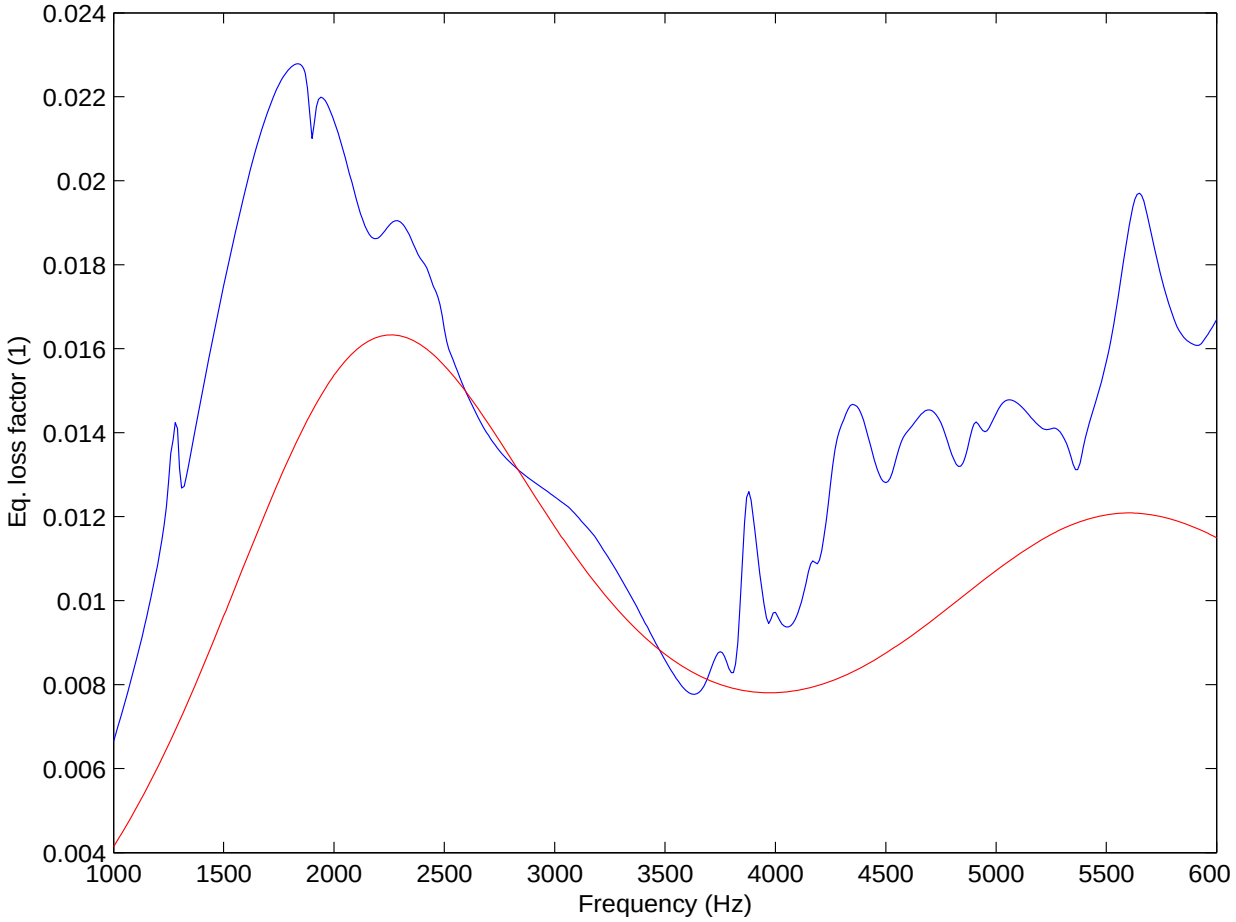
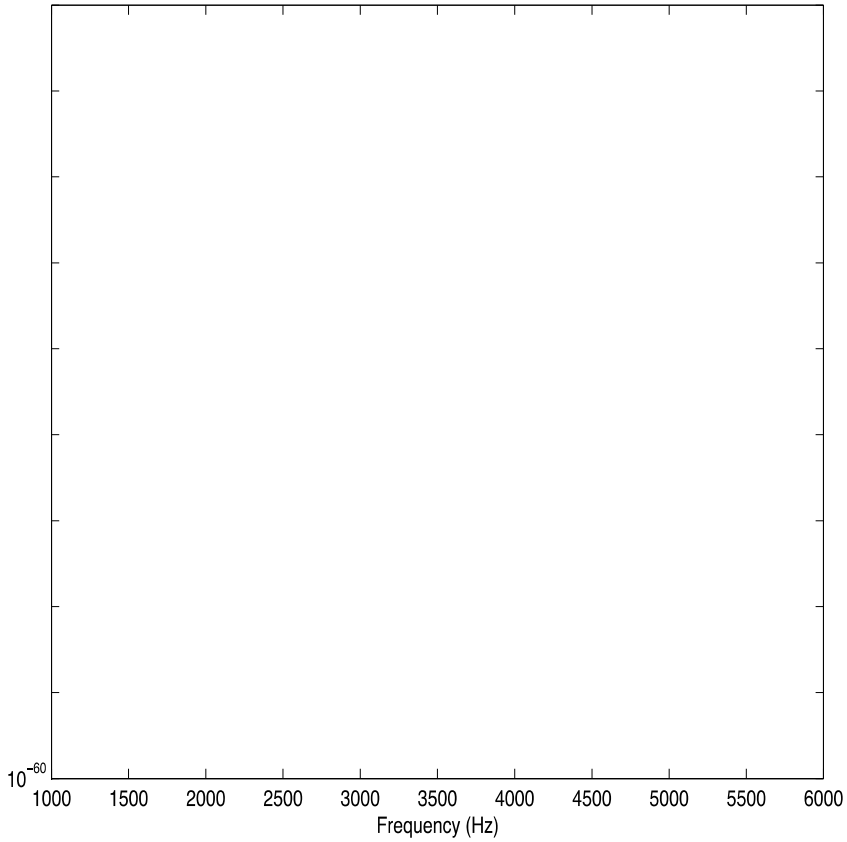
Lossfactor 0.01

Dissipated energy/Kinetic energy



Response spectrum



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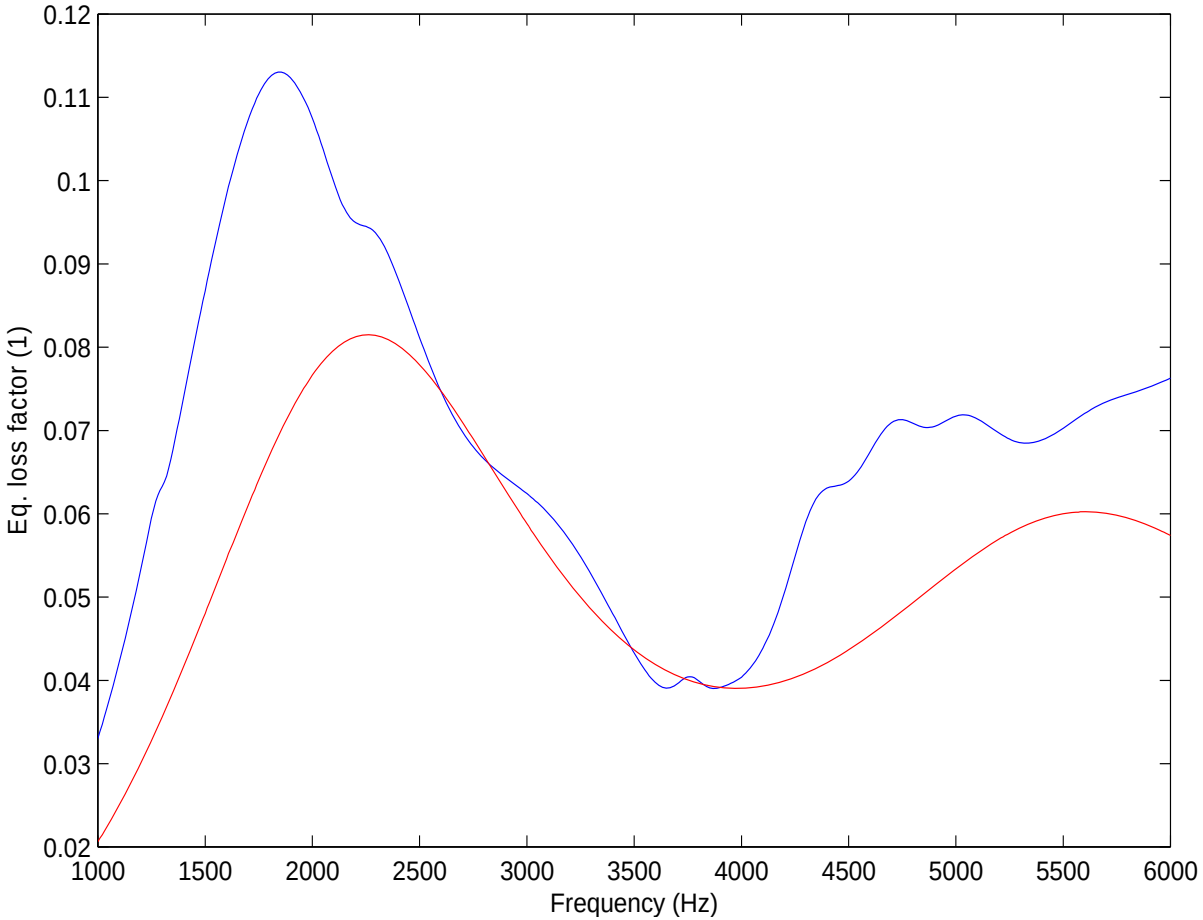
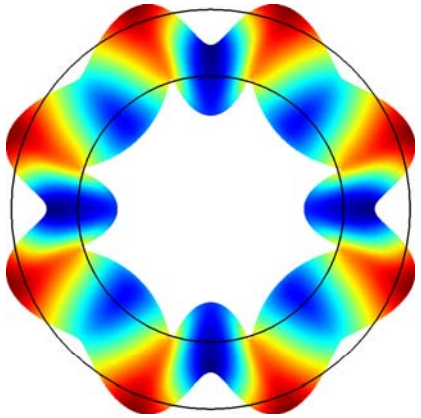
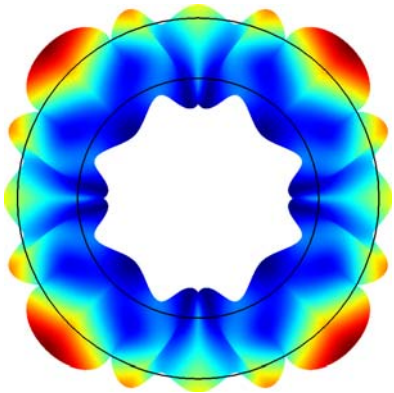
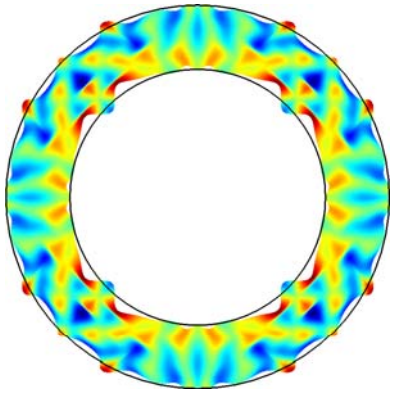
Lossfactor 0.05

Dissipated energy/Kinetic energy



Response spectrum

Frequency (Hz)

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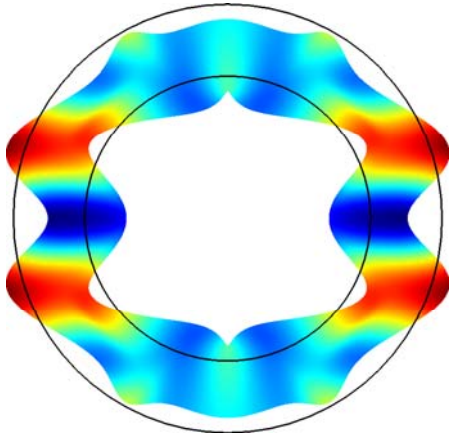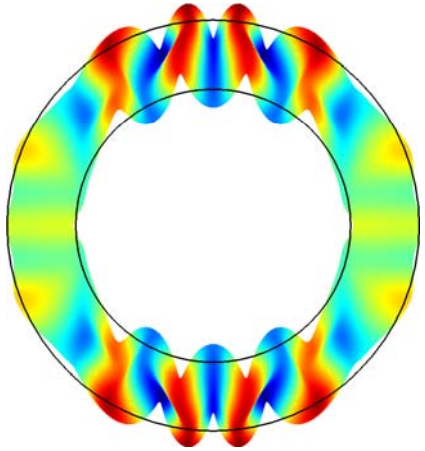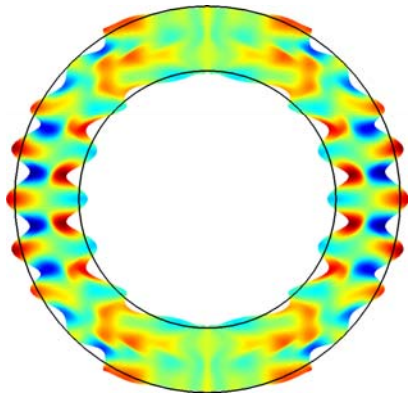
Thank you!