Wave propagation and dispersion in fluid-saturated rigid-framed porous media: checking a new nonlocal macroscopic theory

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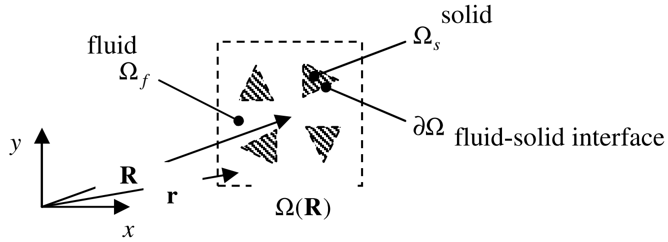
ABSTRACT

Long-wavelength wave propagation and dispersion in fluid-saturated rigid-framed porous media is generally described using the Equivalent-fluid *local* theory [1]. Working for simplicity in 2D, we may sketch as follows one unit periodic cell Ω of one isotropic material, centred at position $\mathbf{R}=(X,Y)$:



Let $I(\mathbf{r})=I(x,y)$ be the fluid indicator function (1 in the fluid, 0 in the solid). Let V and ϕV be the volume of Ω and Ω_f respectively, ϕ being the porosity. Let $\mathbf{v}(t,\mathbf{r})$ and $p(t,\mathbf{r})$ be the fluid velocity and fluid pressure. Then, the macroscopic fluid velocity $\mathbf{V}(t,\mathbf{R})$ and macroscopic fluid pressure $P(t,\mathbf{R})$ are defined according to

$$\mathbf{V}(t,\mathbf{R})=\langle \mathbf{v}\rangle=\frac{1}{V}\int_{\Omega(\mathbf{R})} d\mathbf{r}\; I(\mathbf{r})\mathbf{v}(t,\mathbf{r}) \qquad P(t,\mathbf{R})=\overline{p}=\frac{1}{\phi V}\int_{\Omega(\mathbf{R})} d\mathbf{r}\; I(\mathbf{r})p(t,\mathbf{r}) \tag{1-2}$$

We limit ourselves to small-amplitude long-wavelength wave propagation along axis x. The dependence on $\mathbf{R}$ is a dependence on X and only the component x of the velocity $\mathbf{V}$ – say U – is non vanishing. According to the Equivalent-fluid local theory, the variables $U(t,X)$ and $P(t,X)$ are governed by the macroscopic Eqs

$$\frac{\partial \hat{\rho} U}{\partial t}=-\frac{\partial \phi P}{\partial X}, \qquad \frac{\partial \phi \hat{\chi} P}{\partial t}=-\frac{\partial U}{\partial X} \tag{3-4}$$

with $\hat{\rho}$ and $\hat{\chi}$ being density and compressibility operators defined by kernels functions $\rho(t)$ and $\chi(t)$, i.e.

$$\hat{\rho}U(t,X)=\int_0^\infty dt'\, \rho(t-t')U(t',X), \quad \hat{\chi}P(t,X)=\int_0^\infty dt'\, \chi(t-t')P(t',X) \tag{5-6}$$

In harmonic regime, $U(t,X)=\mathrm{Re}\{\tilde{U}(X)e^{-i\omega t}\}$, $P(t,X)=\mathrm{Re}\{\tilde{P}(X)e^{-i\omega t}\}$, and the Eqs (3-6) yield

$$-i\omega\tilde{\rho}(\omega)\tilde{U}=-\frac{\partial \phi \tilde{P}}{\partial X}, \qquad -i\omega\phi\tilde{\chi}(\omega)\tilde{P}=-\frac{\partial \tilde{U}}{\partial X} \tag{7-8}$$

where $\tilde{\rho}(\omega)$ and $\tilde{\chi}(\omega)$ are the complex Fourier amplitudes of the kernels functions $\rho(t)$ and $\chi(t)$. They are complex-valued densities and compressibilities that can be computed from microgeometry (and fluid parameters) by solving and averaging two independent "cell problems" specifying, resp., the response of the fluid subject to a spatially uniform, time variable - $e^{-i\omega t}$ - bulk force, and, the response of the fluid subject to a spatially uniform, time variable - $e^{-i\omega t}$ - bulk heating. At real frequency ω, there is one wave propagating and attenuating with effective wave number $k(\omega)=\omega\sqrt{\tilde{\rho}(\omega)\tilde{\chi}(\omega)}$, i.e. complex wavespeed

$$c(\omega)=\omega/k(\omega)=\left(\tilde{\rho}(\omega)\tilde{\chi}(\omega)\right)^{-1/2} \tag{9}$$

In spite of its widespread use in acoustic literature, this theory is limited to frequencies and microgeometries ensuring that the medium reacts as a local medium. Dissatisfied with this limitation, and considering that our wave propagation problems may escape a description in terms of the traditional two-scale homogenization theory [2], we have recently proposed a general Equivalent-fluid *nonlocal* theory inspired by electromagnetic considerations [3]. In this new nonlocal theory, direct account is made of the fact that the properties of the medium depend, in general, not only on the time variations of the macroscopic fields (frequency dispersion), but also on their space variations (spatial dispersion). The macroscopic velocity is defined as before (see Eq (1)). The macroscopic pressure is defined in a new manner. Because it reminds of the notion of macroscopic Maxwell strength field $\mathbf{H}$, we denote it by the letter H. Considering, as above, that wave propagation occurs along the axis x, the macroscopic pressure $H(t,X)$ is defined so that

$$\langle p(t,\mathbf{r})\mathbf{v}(t,\mathbf{r})\rangle = H(t,X)\langle \mathbf{v}(t,\mathbf{r})\rangle \qquad (10)$$

This is an acoustic "Poynting-Heaviside" relation. The variables $U(t,X)$ and $H(t,X)$ now obey

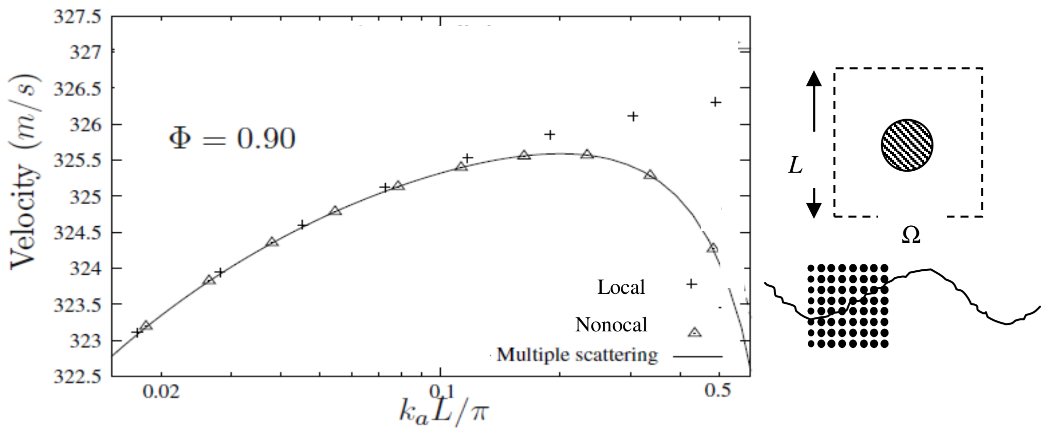
$$\frac{\partial \hat{\rho} U}{\partial t} = -\frac{\partial \phi H}{\partial X}, \qquad \frac{\partial \phi \hat{\chi} H}{\partial t} = -\frac{\partial U}{\partial X} \qquad (11\text{-}12)$$

with $\hat{\rho}$ and $\hat{\chi}$ being density and compressibility operators defined by kernels $\rho(t,X)$ and $\chi(t,X)$, i.e.

$$\hat{\rho}U(t,X) = \int_0^\infty dt' \int dX' \rho(t-t', X-X')U(t',X'), \quad \hat{\chi}H(t,X) = \int_0^\infty dt' \int dX' \chi(t-t', X-X')P(t',X') \qquad (13\text{-}14)$$

The kernel function $\rho(t,X)$ may be written $\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{\rho}(\omega,k) e^{-i\omega t + ikX}$ and likewise for $\chi(t,X)$. Thanks to the definition (10), the complex-valued effective quantities $\tilde{\rho}(\omega,k)$ and $\tilde{\chi}(\omega,k)$ may now be computed from microgeometry by solving and averaging two independent "cell problems" specifying, resp., the response of the fluid subject to a time variable - $e^{-i\omega t}$ - and spatially variable - e^{ikx} - bulk force, and, the response of the fluid subject to a time variable - $e^{-i\omega t}$ - and spatially variable - e^{ikx} - bulk heating. At real frequency ω, there may be more than one wave propagating and attenuating. These waves *i=1,2...* have effective complex wave numbers $k_i(\omega)$, solutions to the dispersion equation $\tilde{\rho}(\omega,k)\tilde{\chi}(\omega,k)\frac{\omega^2}{k^2} = 1$. They have complex wavespeeds $c_i(\omega) = \omega / k_i(\omega)$.

We present here an explicit check of the proposed Equivalent-fluid nonlocal theory, in the special case where the material is a square array of solid cylinders. Indeed, in this case, a direct analytical multiple scattering calculation of the wavespeed $c_1(\omega) = \omega / k_1(\omega)$ of the least attenuated wave is feasible [4]. It may be compared with the wavespeeds calculated with the Equivalent-fluid local and nonlocal theory after solving the two types (local and nonlocal) of "cell problems" (with Freefem++, in the present work). For a medium of porosity $\phi = 0.9$, we plot below the real part of the three wavespeeds (with $k_a \equiv \omega / c_0$ and $L \equiv$ cell dimension).



As illustrated, the Equivalent-fluid local theory predict wavespeeds which are rapidly in error when the wavelengths reduce. This is not the case of the Equivalent-fluid nonlocal theory, which provides a clear validation of its principles. Frequencies at which the nonlocal theory eventually ceases to be valid are frequencies at which the propagation no longer is susceptible of a complete macroscopic description. In the future, it will be interesting to test the nonlocal theory in geometries with resonators – Cf. the ultrasonic metamaterials studied by Fang et al. [5], and contrast it with the traditional two-scale homogenization theory.

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