AN EFFICIENT SOLVER FOR FINITE-ELEMENT POROELASTIC PROBLEMS

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1. Introduction

Assembled structures including porous materials are commonly used in many engineering applications in order to dissipate acoustical or mechanical energy (sound absorption, sound insulation, damping) [1, 4, 6]. In these structures, damping is often due to the inner dissipation mechanisms of the porous material and the optimization of noise control solutions based on the use of such materials requires the development of robust predicting tools.

The development of adapted techniques which can reduce the computational cost of the problem to be solved is then of the utmost importance. Different solutions have still been investigated [9, 10, 8, 7, 5].

2. Theroretical part

Finite-element implementation of $\{\mathbf{u}^s, P\}$ formulations are presented in references [3, 2]:

$$\left(\begin{bmatrix} \hat{P}[\mathbf{K}] & -\widetilde{\gamma}[\mathbf{C}] \\ [\mathbf{0}] & \dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{H}] \end{bmatrix} - \omega^2 \begin{bmatrix} \widetilde{\rho}[\mathbf{M}] & [\mathbf{0}] \\ \widetilde{\gamma}[\mathbf{C}]^t & \dfrac{1}{\widetilde{K}_{eq}}[\mathbf{Q}] \end{bmatrix} \right) \begin{Bmatrix} \mathbf{u}^s \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_s \\ \mathbf{F}_P \end{Bmatrix}. \tag{1}$$

$\hat{P}$, $\widetilde{K}_{eq}$ are respectively the bulk modulus of the solid in-vacuo and of the equivalent fluid associated to the porous material. $\widetilde{\rho}$ is the apparent solid density and $\widetilde{\rho}_{eq}$ is the density of the equivalent fluid model. $\widetilde{\gamma}$ is a coupling coefficient. $[\mathbf{K}]$, $[\mathbf{M}]$, $[\mathbf{C}]$, $[\mathbf{H}]$ and $[\mathbf{Q}]$ are real matrices associated to the finite-element discretization of the spatial operators. Note that these real matrices are not frequency dependent and only depend on the nature of the differential operators. Frequency dependance of system (1) is thereby due to the porous inertial and stiffness coefficients.

Two different generalized eigenproblems are considered: $\{[\mathbf{K}], [\mathbf{M}]\}$ and $\{[\mathbf{H}], [\mathbf{Q}]\}$. These four matrices are real and symmetric and positive. $[\mathbf{M}]$ and $[\mathbf{Q}]$ are also definite. It is then ensured that the eigenvectors and eigenvalues associated to these spectral problems are all real and that the eigenvalues are positive.

A truncation of each modal family need to be performed to warranty the performance the resolution process. Hence, contribution of non selected modes should be accounted for in order to minimize truncation errors. Let $[\mathbf{\Phi}_s]_{n_u \times m_s}$ (resp. $[\mathbf{\Phi}_P]_{n_P \times m_P}$) be the matrix of the first (including rigid-body) m_s (resp. m_P) modes of the generalized eigenvalue problem associated to matrices $\{[\mathbf{K}], [\mathbf{M}]\}$ (resp. $\{[\mathbf{H}], [\mathbf{Q}]\}$). Let $[\mathbf{\Phi}'_s]_{n_u\times(n_u-m_s)}$ (resp. $[\mathbf{\Phi}'_P]_{n_P\times(n_P-m_P)}$) be the matrix of the higher modes. Let $[\mathbf{k}_s^2]$ and $[\mathbf{r}_s^2]$ (resp. $[\mathbf{k}_P^2]$ and $[\mathbf{r}_P^2]$) be the diagonal matrices of the preserved and higher eigenvalues. System (1) is now projected on this decoupled modal basis:

(2)
$$\begin{bmatrix} \hat{P}[\mathbf{k}_s^2]-\omega^2\widetilde{\rho}[\mathbf{I}] & [\mathbf{0}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{kk}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{kh}] \\ [\mathbf{0}] & \hat{P}[\mathbf{r}_s^2]-\omega^2\widetilde{\rho}[\mathbf{I}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{hk}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{hh}] \\ -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{kk}]^t & -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{hk}]^t & \dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{k}_P^2]-\dfrac{\omega^2}{\widetilde{K}_{eq}}[\mathbf{I}] & [\mathbf{0}] \\ -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{kh}]^t & -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{hh}]^t & [\mathbf{0}] & \dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{r}_P^2]-\dfrac{\omega^2}{\widetilde{K}_{eq}}[\mathbf{I}] \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{h}_s \\ \mathbf{q}_P \\ \mathbf{h}_P \end{Bmatrix} = \begin{Bmatrix} \mathbf{\Phi}_s^t\mathbf{F}_s \\ \mathbf{\Phi}_s^{'t}\mathbf{F}_s \\ \mathbf{\Phi}_P^t\mathbf{F}_P \\ \mathbf{\Phi}_P^{'t}\mathbf{F}_P \end{Bmatrix}.$$

$\mathbf{q}_s$ and $\mathbf{h}_s$ (resp. $\mathbf{q}_P$ and $\mathbf{h}_P$) are the contributions of the preserved and higher modes and one has:

$$\mathbf{u}_s = [\mathbf{\Phi}_s]\mathbf{q}_s + [\mathbf{\Phi}'_s]\mathbf{h}_s, \quad \mathbf{P} = [\mathbf{\Phi}_P]\mathbf{q}_P + [\mathbf{\Phi}'_P]\mathbf{h}_P. \tag{3}$$

Projection of the coupling matrix $[\mathbf{C}]$ reads:

$$\begin{bmatrix} [\mathbf{\Gamma}_{kk}] & [\mathbf{\Gamma}_{kh}] \\ [\mathbf{\Gamma}_{hk}] & [\mathbf{\Gamma}_{hh}] \end{bmatrix} = \left[\mathbf{\Phi}_s\,|\mathbf{\Phi}'_s\right]^t[\mathbf{C}]\left[\mathbf{\Phi}_P\,|\mathbf{\Phi}'_P\right]. \tag{4}$$

Systems (1) and (2) are equivalent as they are the representation of the problem in nodal and modal coordinates.

Approximations are now made on modal system (2). First one is to neglect inertial terms for solid and pressure higher modes. (e.g. it means that the excitation pulsation is sufficiently low to have $\hat{P}[\mathbf{r}_s^2] >> \omega^2\widetilde{\rho}[\mathbf{I}]$ and $\dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{k}_P^2] >> \dfrac{\omega^2}{\widetilde{K}_{eq}}[\mathbf{I}]$.). The second approximation is to neglect the higher modes inter coupling term $(-\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{hh}]^t)$ in the block-equation associated to pressure higher modes. System (2) now reads:

(5)
$$\begin{bmatrix} \hat{P}[\mathbf{k}_s^2]-\omega^2\widetilde{\rho}[\mathbf{I}] & [\mathbf{0}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{kk}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{kh}] \\ [\mathbf{0}] & \hat{P}[\mathbf{r}_s^2] & -\widetilde{\gamma}[\mathbf{\Gamma}_{hk}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{hh}] \\ -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{kk}]^t & -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{hk}]^t & \dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{k}_P^2]-\dfrac{\omega^2}{\widetilde{K}_{eq}}[\mathbf{I}] & [\mathbf{0}] \\ -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{kh}]^t & [\mathbf{0}]^t & [\mathbf{0}] & \dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{r}_P^2] \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{h}_s \\ \mathbf{q}_P \\ \mathbf{h}_P \end{Bmatrix} = \begin{Bmatrix} \mathbf{\Phi}_s^t\mathbf{F}_s \\ \mathbf{\Phi}_s^{'t}\mathbf{F}_s \\ \mathbf{\Phi}_P^t\mathbf{F}_P \\ \mathbf{\Phi}_P^{'t}\mathbf{F}_P \end{Bmatrix}.$$

The following step of the proposed method is to condense contributions $\mathbf{h}_s$ and $\mathbf{h}_P$. Not that there is no additional approximations so that all the following developments will leave to systems equivalent to (5). By the way of the last equation-block , it is possible to find the pressure $\mathbf{P}_h$ associated to non-preserved modes:

$$\mathbf{P}_h = [\mathbf{\Phi}'_P]\mathbf{h}_P \approx \widetilde{\rho}_{eq}(\underbrace{[\mathbf{\Phi}'_P][\mathbf{r}_P^2]^{-1}\mathbf{\Phi}_P'^t\mathbf{F}_P}_{\mathbf{\Psi}_P} + \widetilde{\gamma}\omega^2 \underbrace{[\mathbf{\Phi}'_P][\mathbf{r}_P^2]^{-1}\mathbf{\Phi}_P'^t[\mathbf{C}]^t[\mathbf{\Phi}_s]}_{[\mathbf{\Xi}_s]}\mathbf{q}_s). \tag{6}$$

With respect to the frequency dependent coefficient $\widetilde{\rho}_{eq}$, $\mathbf{\Psi}_P$ is the response of higher modes to excitation $\mathbf{F}_P$. This vector is real and frequency independent. $[\mathbf{\Xi}_s]_{n_P\times m_s}$ is the matrix of the response of higher modes to the m_s unitary excitations $[\mathbf{C}]^t[\mathbf{\Phi}_s]$. Hence, the coupling terms associated to non-preserved solid modes are interpreted as forces on the fluid part. $\mathbf{P}_h$ can be introduced in the second block-equation (5):

$$\hat{P}[\mathbf{r}_s^2]\mathbf{h}_s \approx \mathbf{\Phi}_s'^t\mathbf{F}_s + \widetilde{\gamma}\mathbf{\Phi}_s'^t[\mathbf{C}]\mathbf{\Phi}_P\,\mathbf{q}_P + \widetilde{\gamma}\mathbf{\Phi}_s'^t[\mathbf{C}]\mathbf{P}_h. \tag{7}$$

With equation (6), displacement $\mathbf{u}_h$ of non preserved modes can be expressed as a function of the contribution of solid and pressure preserved modes:

$$\mathbf{u}_h = [\mathbf{\Phi}'_s]\mathbf{h}_s \approx \frac{1}{\hat{P}}(\mathbf{\Psi}_s + \widetilde{\gamma}[\mathbf{\Xi}_P]\,\mathbf{q}_P + \widetilde{\rho}_{eq}\mathbf{\Psi}'_s + \widetilde{\rho}_{eq}\widetilde{\gamma}\omega^2[\mathbf{\Xi}'_s]\mathbf{q}_s) \tag{8}$$

with

$$\mathbf{\Psi}_s = [\mathbf{\Phi}'_s][\mathbf{r}_s^2]^{-1}\mathbf{\Phi}_s'^t\mathbf{F}_s, \quad [\mathbf{\Xi}_P] = [\mathbf{\Phi}'_s][\mathbf{r}_s^2]^{-1}\mathbf{\Phi}_s'^t[\mathbf{C}]\mathbf{\Phi}_P \tag{9}$$

and

$$\mathbf{\Psi}'_s = [\mathbf{\Phi}'_s][\mathbf{r}_s^2]^{-1}\mathbf{\Phi}_s'^t[\mathbf{C}]\mathbf{\Psi}_P, \quad [\mathbf{\Xi}'_s] = [\mathbf{\Phi}'_s][\mathbf{r}_s^2]^{-1}\mathbf{\Phi}_s'^t[\mathbf{C}][\mathbf{\Xi}_s]. \tag{10}$$

Hence, system (5) is equivalent to:

$$\begin{bmatrix} \hat{P}[\mathbf{k}_s^2] - \omega^2\widetilde{\rho}[\mathbf{I}] & -\widetilde{\gamma}[\mathbf{\Gamma}_{kk}] \\ -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{kk}]^t & \dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{k}_P^2] - \dfrac{\omega^2}{\widetilde{K}_{eq}}[\mathbf{I}] \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{q}_P \end{Bmatrix} = \begin{Bmatrix} \mathbf{\Phi}_s^t\mathbf{F}_s + \widetilde{\gamma}[\mathbf{\Phi}_s]^t[\mathbf{C}]\mathbf{P}_h \\ \mathbf{\Phi}_P^t\mathbf{F}_P + \widetilde{\gamma}\omega^2[\mathbf{\Phi}_P]^t[\mathbf{C}]^t\mathbf{u}_h \end{Bmatrix}. \tag{11}$$

and the solid and pressure modes contributions $\mathbf{q}_s$ and $\mathbf{q}_P$ are solution of the following system:

(12)

$$\begin{bmatrix} \hat{P}[\mathbf{k}_s^2] - \omega^2\widetilde{\rho}[\mathbf{I}] - \widetilde{\rho}_{eq}\widetilde{\gamma}^2\omega^2[\mathbf{\Phi}_s]^t[\mathbf{C}][\mathbf{\Xi}_s] & -\widetilde{\gamma}[\mathbf{\Gamma}_{kk}] \\ -\widetilde{\gamma}\omega^2[\mathbf{\Gamma}_{kk}]^t - \dfrac{\widetilde{\gamma}\omega^2\widetilde{\rho}_{eq}^2\omega^2}{\hat{P}}[\mathbf{\Phi}_P]^t[\mathbf{C}]^t[\mathbf{\Xi}'_s] & \dfrac{1}{\widetilde{\rho}_{eq}}[\mathbf{k}_P^2] - \dfrac{\omega^2}{\widetilde{K}_{eq}}[\mathbf{I}] - \dfrac{\widetilde{\gamma}^2\omega^2}{\hat{P}}[\mathbf{\Phi}_P]^t[\mathbf{C}]^t[\mathbf{\Xi}_P] \end{bmatrix} \begin{Bmatrix} \mathbf{q}_s \\ \mathbf{q}_P \end{Bmatrix}$$

(13)

$$= \begin{Bmatrix} \mathbf{\Phi}_s^t\mathbf{F}_s + \widetilde{\gamma}[\mathbf{\Phi}_s]^t[\mathbf{C}]\widetilde{\rho}_{eq}\mathbf{\Psi}_P \\ \mathbf{\Phi}_P^t\mathbf{F}_P + \dfrac{\widetilde{\gamma}\omega^2}{\hat{P}}[\mathbf{\Phi}_P]^t[\mathbf{C}]^t(\mathbf{\Psi}_s + \widetilde{\rho}_{eq}\mathbf{\Psi}'_s) \end{Bmatrix}.$$

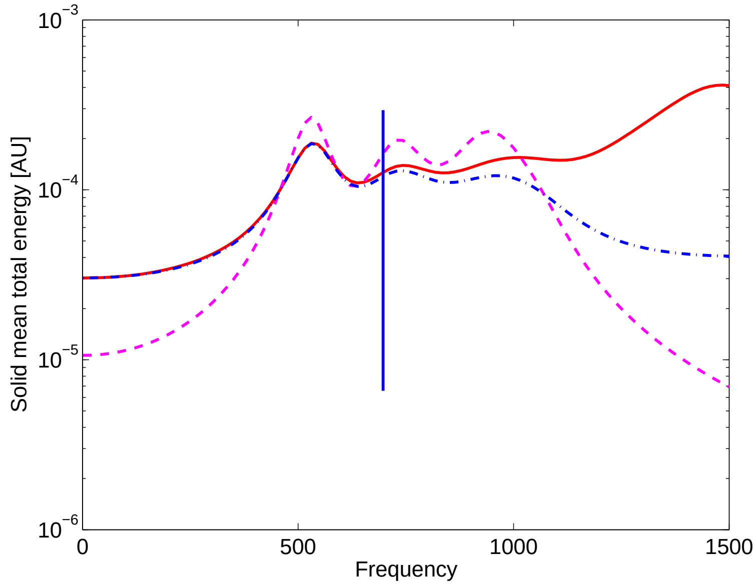

Figure 1. Compression excitation. Red: Reference; Magenta: Decoupled modal analysis; Blue: proposed approch

3. Result

Figure 1 and 2 present results obtained for a 2D problem. A melamine foam of dimension 4 centimeters in z direction and 40 centimeters along x. It is excited on the solid phase in the long dimension. Two excitations are considered. The first one is along z (i.e. mainly compression) and the second one is along x (i.e. mainly shear. 5 modes are selected for each phase. The method shows good agreement of the technique with the reference solution. More results will be presented in the conference.

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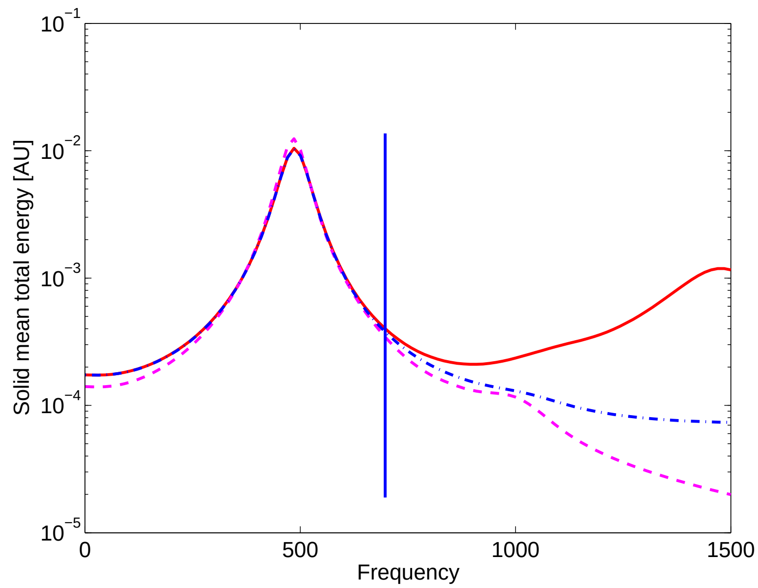

FIGURE 2. Shear excitation. Red: Reference; Magenta: Decoupled modal analysis; Blue: proposed approch

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